Credit Spreads in Illiquid Markets: Model and Implementation

Gonzalo Cortazar, Eduardo S. Schwartz, and Claudio Tapia

ABSTRACT: This paper presents a methodology for estimating a family of credit spread term structures in a market with few transactions. The authors propose partitioning the market into risk classes and modeling credit spread term structures for each risk class using a multifactor Vasicek model with some common and some risk class-specific factors. The approach uses information on the cross section and time series of corporate bonds in all the risk classes to estimate the term structure of credit spreads in each risk class. The model is jointly estimated using an extended Kalman filter and implemented using Chilean corporate and government bonds.

KEY WORDS: bond spreads, emerging markets, Kalman filter.

Credit risk accounts for the possibility of loss due to a debtor's failure to repay principal or interest in a timely manner. This counterparty risk, also called "default risk," together with liquidity risk, explains why securities issued with similar promised cash trade at a yield above the risk-free interest rate at which liquid government bond issues are priced. The credit risk literature attempts to explain the yield differential between risky and risk-free securities and to estimate a term structure of credit spreads, which can be used to price risky assets that are not traded.

When modeling credit spreads, two issues need to be addressed. First, in many markets there are extremely few observations of risky bond transactions and default events. This dramatically complicates the calibration of models and often results in estimated spreads inconsistent with default frequencies observed in the data (Huang and Huang 2003; Jones et al. 1984). The second issue a model of credit spreads must account for is the dynamics of these spreads. For example, during a financial crisis there is a considerable decrease in demand for instruments subject to credit risk, which triggers a process known as "flight to quality." This increase in demand for risk-free instruments relative to risky securities induces a significant increase in credit spreads. Adequately accounting for this spread-related dynamic becomes a challenge for credit spreads models.

There are two main approaches to modeling credit spreads.[1] The first one uses structural models (Merton 1974) and considers a company's bonds and stocks as contingent claims on its assets. In structural models, default is triggered when the face value of debt at maturity (initially modeled as a single instrument) is smaller than the firm's asset value. This approach is useful when the model intends to find the determinants of spreads, but the empirical results, obtained using historical rates of default, generate much smaller

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spreads than those observed in the market. This problem is known as the credit spread puzzle (Eom et al. 2004; Ericsson et al. 2006).

Extensions to this literature are less restrictive in their assumptions than Merton (1974) was: Collin-Dufresne and Goldstein (2001) make the firm's debt level change over time with reversion to a long-term mean; Duffie and Lando (2001) include imperfect information to explain high corporate bond spreads; and Longstaff and Schwartz (1995) incorporate stochastic interest rates and modify some boundary conditions to achieve a better fit to observed prices.

The other main approach to modeling spreads uses reduced-form models, also called intensity models. These dynamic models usually require less information and generate term structures without explicitly modeling the determinants of credit spreads. This approach values risky assets by discounting their cash flows at a rate higher than the risk-free rate. The spread added to the risk-free rate is determined by the intensity of a Poisson process, modeling the arrival of the first jump, which represents default. Duffie and Singleton (1999) show that under certain regularity conditions, these models may be estimated by the same tools used in modeling risk-free term structures, even when factors follow a dynamic jump diffusion process. A major advantage of these models is that they are able to generate credit spreads more in line with those observed in the market. Nevertheless, they lack a structural definition of default and therefore are not very useful for finding an economic interpretation of the actual spreads.

Additional related research includes Driessen (2005), who proposes a model that has jumps for the spread of corporate bonds and, using firm-specific factors and factors common to all firms, shows that the former have a larger impact in the observed prices; Duffee (1999), who, in modeling the default probability of individual firms, allows for correlations between them and the risk-free interest rate process; Liu et al. (2006), who present a model that jointly estimates the dynamics of interest rate swap spreads and the implied risk-free term structure, finding negative correlations between them; Liu et al. (2007), who use corporate bond data to estimate the effect of personal taxes on bond spreads; and Longstaff et al. (2005), who use a reduced-form model to measure the effect of default and nondefault components on corporate spreads. In an emerging market, the lack of liquidity makes the problem of finding credit spreads even more difficult. First, not only corporate bonds, but also government bonds are illiquid in these markets, which makes finding the risk-free benchmark to measure spreads an additional issue. This makes static models, like those of Nelson and Siegel (1987) and Svensson (1994) and which are broadly used in modeling term structures of interest rates in developed economies, a bad choice for emerging markets (Cortazar et al. 2007). Second, corporate bonds are so illiquid in these markets that estimating individual spreads by issuer is a daunting task, it being essential to pool the available information. One of the most common groupings in this literature is by credit rating. Godlewski (2007) highlights the role of ratings in emerging economies, where underdevelopment of financial markets make agency ratings a valuable source of information. Although Duffee (1999) found that the levels and dynamics of hazard rates depend on the rating modeled, Feldhutter and Lando (2008) propose a rating-based model, arguing that it reduces measurement errors, thereby making the estimation of relationships among rating and swap spreads less noisy. This paper proposes and estimates a dynamic model for the rates of risky bonds that uses current and historical information on corporate and government bonds to obtain spread term structures. A two-step estimation approach is proposed. First, we use the methodology proposed in Cortazar et al. (2007) for risk-free term structure estimation to deal with incomplete data panels and model the risk-free rate as a generalized three-factor Vasicek process. Then, the risky bond data is grouped into different risk classes with spreads modeled as a sum of some common and some risk class-specific factors. The term structures of all risk classes are jointly estimated using an extended Kalman filter.[2] The model is implemented and tested using a sample of Chilean treasury and corporate bonds, but the approach may be used in any other market where lack of information is a relevant feature.

The model could easily be extended to consider credit derivatives such as credit default swaps (CDS), which, when available, tend to be more liquid than corporate bonds. However, CDS markets are not common in emerging markets. Thus in our implementation we restrict ourselves to the bond market.

The main contribution of the proposed approach is to develop a methodology that leverages information from the risk-free term structure and the historical correlations between the different risk classes to overcome the lack of transactions typical of an emerging market, providing a framework to estimate the prices of individual risky securities in an incomplete data set. By using unobserved latent factors, we restrict the number of variables involved in the model while maintaining high explanatory power in our estimations, as evidenced by our in-sample and out-of-sample results. Although the model relies on rating classes to estimate term structures, it allows for deviations from the observed spreads.

In essence, the key feature of our approach is that it uses information on both the cross section and the time series of corporate bonds in all risk classes to estimate the term structure of credit spreads in each risk class.

The Risk-Free Term Structure in Illiquid Markets

The estimation of a risk-free term structure is of crucial importance when credit spreads are measured. In an ideal world, there would be an equivalent government bond for every risky bond for which a credit spread needs to be measured. Unfortunately, this is not normally the case. Therefore, it is necessary to specify a model for the risk-free term structure that provides estimations for the risk-free interest rates across all maturities.

Most of the literature suggests estimating the current term structure of interest rates using a parametric or nonparametric static model that fits current bond prices or yields quoted in the market (Fisher et al. 1995; Nelson and Siegel 1987; Svensson 1994). These models have been successfully implemented for several developed markets, usually providing simple and well-fitting estimations.

Developed markets typically have many price observations for each day, and they are fairly well distributed across all maturities. This is not the case in emerging markets, where sparse data makes the use of these static models unreliable because, among other things, they often overestimate the term structure of volatilities (Cortazar et al. 2007).

To solve this problem Cortazar et al. (2007) propose the use of dynamic models, which combine current and historical information to jointly estimate the term structure of the spot rate and its dynamics using no-arbitrage arguments. By doing so, they argue, market information is extracted from current and historical data. This allows for more reliable estimations of the term structure, based on the calibration of one of the many dynamic models proposed in the literature (Brennan and Schwartz 1979; Cox et al. 1985; Vasicek 1977).

In this paper we use the methodology proposed by Cortazar et al. (2007) to model and calibrate the risk-free term structure in an emerging market with few transactions. The authors are interested in the real risk-free rates obtained from transactions of inflation-protected government bonds and use a general N-factor stochastic model in which each of the N factors follows a mean-reverting Vasicek process, as in Langetieg (1980). This model, as opposed to others such as Cox–Ingersoll–Ross (CIR) type of models, allows for negative rates, which might be desirable when modeling real rates.

Let r_t be the real spot risk-free rate, then

$$r_t = 1' x_t^r + \delta_0^r, \tag{1}$$

where x_t^r ($N^r \times 1$) is a vector of N^r state variables. The vector x_t^r follows a dynamic process given by a stochastic differential equation (SDE) (2):

$$dx_t^r = -K x_t^r dt + \Sigma dw_t^r, \tag{2}$$

where $K(N^r \times N^r)$ and $\Sigma(N^r \times N^r)$ are diagonal matrices and the elements κ_{ii} and σ_{ii}^2 represent the mean reversion rate and variance of factor x_i, respectively. As this is a stationary model, each element of matrix K must be nonzero. $dw_t^r(N^r \times 1)$ is an array of Brownian motion increments such that

$$E\left[w_t^r w_t^{r\prime} \right] = \Omega^{\mathbf{r}} t. \tag{3}$$

In Equation (3), each element of matrix $\Omega^{\mathbf{r}}$ ($N^r \times N^r$), ρ_{ij}^r corresponds to the instantaneous correlation between factors i and j. SDE (2) specifies no long-run mean for the state variables, implying that each stochastic factor reverts to zero, making the instantaneous interest rate r revert to a long-term mean given by δ_0^r under the physical probabilities.

Assuming the existence of an equivalent probability measure Q and a constant market price of risk λ,[3] the risk-adjusted process for factors x^r is

$$dx_t^r = -\left(\lambda + K x_t^r\right) dt + \Sigma dw_t^{Q,r}, \tag{4}$$

where $w_t^{Q,r}$ is the equivalent Brownian motion under the risk-neutral probability measure. Under this specification, it is possible to find a closed-form solution for the value of a risk-free discount bond $P(x_t^r, \tau)$ as a function of the process parameters, where x_t^r are the state variables at time t gathering all relevant information up to time t, and τ is the time to maturity of the discount bond. The solution for the partial differential equation (PDE) of a Vasicek model is given by Equation (5):

$$P\left(x_t^r, \tau\right) = e^{\left(u(\tau)' x_t^r + v(\tau)\right)}. \tag{5}$$

Plugging derivatives of $P(x_t, \tau)$ into Vasicek's PDE we find:

$$u_i(\tau) = -\left(\frac{1 - e^{-\kappa_{ii}\tau}}{\kappa_{ii}}\right)$$

$$v(\tau) = \sum_{i=1}^{N^r} \frac{\lambda_i}{\kappa_{ii}} \left(\tau - \frac{1 - e^{-\kappa_{ii}\tau}}{\kappa_{ii}}\right) - \delta_0^r \tau \tag{6}$$

$$+ \frac{1}{2} \sum_{i=1}^{N^r} \sum_{j=1}^{N^r} \frac{\sigma_{ii}\sigma_{jj}\rho_{ij}^r}{\kappa_{ii}\kappa_{jj}} \left(\tau - \frac{1 - e^{-\kappa_{ii}\tau}}{\kappa_{ii}} - \frac{1 - e^{-\kappa_{jj}\tau}}{\kappa_{jj}} + \frac{1 - e^{-\left(\kappa_{ii} + \kappa_{jj}\right)\tau}}{\kappa_{ii} + \kappa_{jj}}\right).$$

The model is a canonical representation in the sense of Dai and Singleton (2000), which can be econometrically identified. Given the price $P(x_t^r, t)$ of a discount bond, we can represent its yield as a function of $u(t)$ and $v(t)$:

$$y\left(x_t^r, \tau\right) = -\frac{1}{\tau}\log\left(P\left(x_t^r, \tau\right)\right) = -\frac{1}{\tau}\left(u\left(\tau\right)' x_t^r + v\left(\tau\right)\right). \tag{7}$$

Note that there is a linear relationship between the observable variable and the Gaussian latent factors, which implies that the discount bond yields are also Gaussian.

Cortazar et al. (2007) show that calibrating this model using an extended Kalman filter with an incomplete panel of prices provides reliable estimates of the real risk-free term structure when applied to inflation-protected Chilean government bonds. Similarly, Babbs and Nowman (1999) find that two- and three-factor generalized Vasicek models estimated with the Kalman filter are successful in modeling zero coupon term structures for interbank rates using U.S. data.

The Credit Spread Model and Estimation Methodology for Illiquid Markets

The Model

The dynamic credit spread model uses the information from risky and risk-free bond prices to estimate the term structures for risky bonds. The risky bond observations are grouped into risk classes, and an estimation methodology is proposed to jointly estimate a family of term structures.

The proposed model is based on Duffie and Singleton (1999), who established that under certain technical conditions, risky bonds might be priced using a discount rate higher than the risk-free interest rate and that this rate may be parameterized using multifactor models just like those used to model risk-free term structures. This framework assumes that under the risk-neutral probability measure Q, there is a hazard rate h_t and a fractional loss in the bond value at default L_t.

The value of a risky bond V_t that pays 1 on time T might therefore be expressed as

$$V_t = E_t^Q\left[\exp\left(-\int_t^T R_s ds\right)\right], \tag{8}$$

where

$$R_t = r_t + h_t L_t, \tag{9}$$

where R_t is the sum of the risk-free rate and an additional term known as the mean-loss rate. Additional terms could be added to model differences in liquidity or taxation, so a more general expression for the rate of a risky bond is

$$R_t = r_t + \underbrace{\sum s_t^i}_{S_t}, \tag{10}$$

where each s_t^i represents a different source of spread due to credit, liquidity, or other risks, which total S_t. In order to face the extremely incomplete data panels of individual risky bonds, we group bonds into risk classes. We then model the spread of each risk class as a function of the risk-free factors plus some common and some risk class-specific factors:

$$S_{t,j} = \gamma_j 1' x_t^r + \delta_j^{c\prime} x_t^c + \delta_j^{g\prime} x_{t,j}^g + \delta_0^j, \tag{11}$$

where $S_{t,j}$ is the spread for risk class j, and γ_j is a constant linking the spread of class j and the state variables that model the risk-free interest rate. This parameter intends to capture the dependence of the spread on the risk-free rate that has been documented for developed markets.[4] Vector $x_t^r(N^r \times 1)$ is an array of state variables modeling the risk-free term structure, $x_t^c(N^c \times 1)$ is a vector of common factors among risk classes, while $x_{t,j}^g(N^{g,j} \times 1)$ represents class-specific factors modeling the portion of the spread specific to the risk class j. Vectors $\delta_j^c(N^c \times 1)$ and $\delta_j^g(N^{g,j} \times 1)$ are arrays of coefficients multiplying the common and class-specific state variables, respectively. Finally, δ_0^j is a scalar parameter.

This model tries to capture the comovements among the different risk classes. The intuition behind it is that economic conditions, such as temporary supply and demand imbalances or economic cycles, tend to affect corporate debt markets as a whole, tightening or widening all the observed spreads. Driessen (2005) proposes a similar structure, finding that common factors are more important than individual factors in observed prices.

These state variables by themselves do not necessarily have a direct economic interpretation: A single state variable or a combination of them might represent a mean-loss rate, a liquidity factor, or any other variable affecting the pricing of risky bonds. Some efforts have been made by other authors to relate latent factors to observable economic variables. For example, Bhar and Handzic (2011) model corporate bond spreads using three latent variables that they relate to equity returns, volatility, and long-term interest rates. In this paper, however, we focus on setting up the model and proposing a useful estimation methodology to deal with scarcity of data in emerging markets, rather than on the interpretation of these state variables.

Given the model for the risk-free term structure and the expression for the spread, the discount rate of a class j risky bond may be defined as

$$R_{t,j} = \left(1 + \gamma_j\right) 1' x_t^r + \delta_j^{c\prime} x_t^c + \delta_j^{g\prime} x_j^g + \delta_j^{R,0}, \tag{12}$$

where the instantaneous rate of a bond classified into risk class j is modeled using N^j factors:

$$N^j = N^r + N^c + N^{g,j}. \tag{13}$$

We assume that common and class-specific factors (x^c and x^g, respectively) follow a generalized Vasicek process under a Q measure similar to the one specified for the risk-free factors (x_t^r) shown in Equation (4):

$$dx_t^c = -\left(\lambda^c + K^c x_t^c\right) dt + \Sigma^c dw_t^{Q,c} \tag{14}$$

$$dx_t^{g,j} = -\left(\lambda^{g,j} + K^{g,j} x_t^g\right) dt + \Sigma^{g,j} dw_t^{Q,g}, \tag{15}$$

where vectors $\lambda^c(N^c \times 1)$ and $\lambda^{g,j}(N^{g,j} \times 1)$ represent constant market prices of risk for common and class j factors, respectively. $K^c(N^c \times N^c)$ and $K^{g,j}(N^{g,j} \times N^{g,j})$ are diagonal matrices of the mean-reverting parameters. $\Sigma^c(N^c \times N^c)$ and $\Sigma^{g,j}(N^{g,j} \times N^{g,j})$ are the diagonal variance matrices for common and class-specific factors. Finally, we allow for correlations between all the Brownian motions (i.e., $w_t^{Q,r}, w_t^{Q,c}, w_t^{Q,g_1}, \ldots, w_t^{Q,g_J}$), where J represents all the risk classes into which we have divided the risky bond observations) such that

$$w_t^Q = \begin{bmatrix} w_t^{Q,r} \\ w_t^{Q,c} \\ w_t^{Q,g_1} \\ \vdots \\ w_t^{Q,g_J} \end{bmatrix} \tag{16}$$

$$E\left[w_t w_t{'}\right] = \Omega t. \tag{17}$$

Defining all the risk factors as Gaussian has the advantage of allowing for an easy application of the extended Kalman filter to estimate the model, while providing closed-form solutions for the price of a risky bond. In addition, Dai and Singleton (2002) argue that Gaussian models are better at capturing the dynamics of the risk premia in affine models. Nevertheless, if both the risk-free and risky term structures were defined using a CIR model, even though the solution would still be closed form and the framework applicable, estimation would be more demanding.[5] Applying Ito's lemma, the risky bond has a representation similar to the default-free bond, as stated in Equations (5) and (6). Hence, using the same procedure we can uncover the terms $u_i^j(\tau)$ and $v^j(\tau)$ for the price of a risky bond in class j:

$$u_i^j(\tau) = -\delta_i^j \left(\frac{1 - e^{-\kappa_{ii}^j \tau}}{\kappa_{ii}^j} \right)$$

$$\begin{aligned} v^j(\tau) &= \sum_{i=1}^{N^j} \delta_i^j \frac{\lambda_i^j}{\kappa_{ii}^j} \left(\tau - \frac{1 - e^{-\kappa_{ii}^j \tau}}{\kappa_{ii}^j} \right) - \delta_j^{R,0} \tau \\ &+ \frac{1}{2} \sum_{i=1}^{N^j} \sum_{k=1}^{N^j} \delta_i^j \delta_k^j \frac{\sigma_{ii}^j \sigma_{kk}^j \rho_{ik}^j}{\kappa_{ii}^j \kappa_{kk}^j} \left(\tau - \frac{1 - e^{-\kappa_{ii}^j \tau}}{\kappa_{ii}^j} - \frac{1 - e^{-\kappa_{kk}^j \tau}}{\kappa_{kk}^j} + \frac{1 - e^{-\left(\kappa_{ii}^j + \kappa_{kk}^j\right)\tau}}{\kappa_{ii}^j + \kappa_{kk}^j} \right), \end{aligned} \tag{18}$$

where $\delta_i^j = (1 + \gamma^j)$ for the n^r state variables modeling the risk-free rate in risk class j, $\delta_i^j = \delta_j^{c,d}$ for the dth common factor in class j, and $\delta_i^j = 1$ for each of the $N^{g,j}$ state variables of the risk class to be modeled. Given the dynamics for each factor, the rate R_j reverts under physical probabilities to a long-term mean, $\delta_j^{R,0}$, that corresponds to the sum of δ_0^r and δ_0^j.

Finally, the volatility term structure for each risk class j is extracted from the variance of the bond price process for different maturities:

$$\sigma_j^2(\tau) = \sum_{i=1}^{N^j} \sum_{k=1}^{N^j} \delta_i^j \delta_k^j \sigma_{ii}^j \sigma_{kk}^j \rho_{ik}^j \left(\frac{e^{-\kappa_{ii}^j \tau} - 1}{\kappa_{ii}^j} \right) \left(\frac{e^{-\kappa_{kk}^j \tau} - 1}{\kappa_{kk}^j} \right). \tag{19}$$

Estimation Methodology: The Extended Kalman Filter

The estimation of the dynamic model requires the determination of the model parameters and the time series of state variables. The Kalman filter methodology allows for the estimation of state variables using current and historical information, while model parameters may be obtained by maximum likelihood.[6] The filter allows for measurement errors in the observations, which is particularly important in emerging markets. Lack of liquidity increases the uncertainty for some observed prices, making it essential to have an estimation methodology that provides a comparison point to decide whether to fully believe an observed price or not. The filter uses historical information to calibrate observation error variances and adjust the estimated term structures accordingly. This feature also helps when proposing a model based on credit ratings, which have considerable inertia and take some time to update. The extended version of the Kalman filter is used to deal with nonlinearities in the state space representation of the problem. A detailed explanation of the Kalman filter may be found in Harvey (1989).

In this paper, we use an adaptation of the Kalman filter that enables its use with incomplete data panels. A description of this extension may be found in Cortazar et al. (2007) and Sørensen (2002).

Joint Estimation Procedure

As mentioned earlier, the Kalman filter allows for the estimation of parameters and state variables in a dynamic model. However, our model requires the estimation of more than one term structure. On one side, we have the risk-free term structure, which is modeled using the approach suggested in Cortazar et al. (2007), and, on the other side, we have each of the risk classes into which the risky bond data is grouped.

Duffee (1999) points out a similar problem in estimating a term structure for several individual firms along with the default-free term structure. He first estimates the state variables for the risk-free curve, then carries out individual estimations for each term structure assuming the risk-free state variables previously estimated are the true values.

We also propose a two-step calibration procedure in which we first estimate the risk-free term structure to deal with liquidity differences between government and corporate bonds. Then we use a joint estimation of the term structures for all risk classes. This procedure shares some similarities to the one used for multicommodity models in Cortazar and Eterovic (2010) and Cortazar et al. (2008).[7]

This two-step procedure allows us to use the information from a relatively more liquid market (government bonds) to estimate the typically very illiquid risky term structures. It also lightens the computational burden by decreasing the number of parameters to be estimated in each step, reducing both the time and complexity involved in finding satisfactory estimations.

In addition, joint estimation at the second step can be used to compute correlations between the factors for different risk classes, which makes it possible for any observation in any risk class to update the estimation for all classes according to the historical correlations between them. This dramatically increases the amount of information available, which is particularly important in low-liquidity emerging markets.

In order to perform the joint estimation, it is necessary to know the functional form of the vectors and matrices involved. It can be shown that the measurement equation for coupon-paying bonds under the extended Kalman filter framework is

$$z_t = y_{(0,t|t-1)} + \bar{H}_t \left(x_t - \hat{x}_{(t|t-1)} \right) + \varepsilon_t. \tag{20}$$

We arrange measurement equation matrices such that they can be represented as follows:

$$z_t = \begin{bmatrix} z_t^r \\ z_t^s \end{bmatrix}, \; y_{(0,t|t-1)} = \begin{bmatrix} y_{(0,t|t-1)}^r \\ y_{(0,t|t-1)}^s \end{bmatrix} \bar{H}_t = \begin{bmatrix} \bar{H}_t^{rr} & \bar{H}_t^{rs} \\ \bar{H}_t^{sr} & \bar{H}_t^{ss} \end{bmatrix}, \; x_t = \begin{bmatrix} x_t^r \\ x_t^s \end{bmatrix},$$
$$\hat{x}_{(t|t-1)} = \begin{bmatrix} \hat{x}_{(t|t-1)}^r \\ \hat{x}_{(t|t-1)}^s \end{bmatrix}, \; \varepsilon_t \sim N(0,\Phi), \Phi = \begin{bmatrix} \Phi^r & 0 \\ 0 & \Phi^s \end{bmatrix}, \tag{21}$$

where $z_t^r(M_t^r \times 1)$ are the M_t^r observed yields of risk-free bonds used to estimate the risk-free term structure at time t, and $z_t^s(M_t^s \times 1)$ are the observed risky yields to estimate the term structures for all the risk classes, where M_t^s is the sum of all risky observations at time t across all risk classes. In addition, $y^r_{(0,t|t-1)}$ and $y^s_{(0,t|t-1)}$ $(M_t^s \times 1)$ are the yield estimations for risk-free and risky bonds obtained at time t using the information up to $t - 1$. $\bar{H}_t^{rr}(M_t^r \times N^r)$ is the transition matrix linking risk-free observations with risk-free state variables while $\bar{H}_t^{rs}(M_t^r x\{N^c + \Sigma_{j=1}^J N^{g,j}\})$ relates risk-free observations to spread factors (common and class-specific state variables). Along the same lines, $\bar{H}_t^{sr}(M_t^s \times N^r)$ and $\bar{H}_t^{ss}(M_t^r x\{N^c + \Sigma_{j=1}^J N^{g,j}\})$ relate risky observations to risk-free and spread factors, respectively. $x_t^r(N^r \times 1)$ are the time t state variables modeling the risk-free term structure, while $x_t^s(\{N^c + \Sigma_{j=1}^J N^{g,j}\} \times 1)$ groups the additional time t state variables modeling the spread (common and class-specific factors). $\hat{x}^r_{(t|t-1)}(N^r \times 1)$ and $\hat{x}^s_{(t|t-1)}(\{N^c + \Sigma_{j=1}^J N^{g,j}\} \times 1)$ are the Kalman filter predictions made using information up to time $t - 1$ for risk-free and risky factors, and $\varepsilon_t(\{M_t^r + M_t^s\} \times 1)$ is the random Gaussian vector representing the measurement errors with mean 0 and diagonal variance matrix $\Phi(\{M_t^r + M_t^s\} \times \{M_t^r + M_t^s\})$, comprising $\Phi^r(M_t^r \times M_t^r)$ and $\Phi^s(M_t^s \times M_t^s)$, which are also diagonal matrices having individual variances in their risk-free and risky observation measurement errors.

The assumption behind the separation into two steps is that risk-free observations provide enough information to estimate the default-free term structure and therefore we can ignore risky bonds in this first step. As a consequence, spread variables x_t^s are also ignored in this step, reducing Equation (20) to

$$z_t^r = y^r_{(0_t|t-1)} + \bar{H}_t^r \left(x_t^r - \hat{x}^r_{(t|t-1)} \right) + \varepsilon_t^r, \tag{22}$$

where $\varepsilon_t^r(M_t^r \times 1)$ is a Gaussian random vector distributed $N(0, \Phi^r)$. For the second step, we use only risky observations and assume that previously estimated risk-free variables are the true ones. Therefore in this step we do not need previous period predictions for these variables, and set $x^r_{(t|t-1)} = x_t^r$. This reduces our second-step measure equation to

$$z_t^s = y^s_{(0_t|t-1)} + \bar{H}_t^s \left(x_t^s - \hat{x}^s_{(t|t-1)} \right) + \varepsilon_t^s. \tag{23}$$

This estimation procedure has the advantage of providing estimations for risky term structures that are consistent with the default-free term structure and take advantage of the correlations between risk classes in order to generate better estimations of their term structures.

Results

To illustrate the methodology, we implement the model using a sample of government and corporate Chilean bonds, traded between January 2003 and December 2006. We then use data for 2007 to test the model out of sample. In this market, around 90 percent of outstanding corporate debt is issued as inflation-protected bonds.[8] Therefore, we choose these inflation-protected bonds in order to estimate real, as opposed to nominal, term structures for both risk-free and risky interest rates. As negative interest rates are permitted when using inflation-protected (real) bonds, the choice of a Vasicek model seems appropriate.

Following Cortazar et al. (2007), we estimate a three-factor generalized Vasicek model for the risk-free term structure. Then, we test our spread model using the simplest specification possible: one common factor and one specific factor for each risk class used.

The Data

The estimation of the risk-free term structure is made using daily transactions, made on the Santiago Stock Exchange from January 2003 to December 2006, of bullet and amortizing inflation-protected bonds issued by both the Treasury and the Central Bank of Chile, an institution equivalent to the U.S. Federal Reserve. Table 1 summarizes the data used to estimate the default free term structure.

The risky term structures are estimated using a sample of Chilean inflation-protected corporate bonds, with no embedded options, traded between January 2003 and December 2006 on the Santiago Stock Exchange. We use only investment grade bonds (i.e., credit ratings ranging from AAA to BBB) with time to maturity of more than one year. The sample is composed of 202 securities from fifty-seven issuers. On average, 2.2 percent of outstanding bonds trade on a given date.

Table 2 summarizes the mean and standard deviation of spreads by credit rating traded during the in-sample period. Note that most of the observations pertain to the AA category and represent more than 50 percent of the risky bonds sample. Note also that the better the rating, the smaller the average spread, which suggests that the rating class is, on average, a good predictor of the category's risk. Table 2 also shows an increasing standard deviation as credit ratings decrease, which is similar to the findings of Delianedis and Geske (2001) for the U.S. market.

Empirical Results

As stated above, our estimation methodology considers the estimation of the risk-free term structure as a first step. Table 3 shows the parameters estimated using the generalized Vasicek model.[9]

Table 3 shows the results of the estimation for the risk-free yield curve. All the parameters are statistically significant, in line with the results obtained in Cortazar et al. (2007). The ϕ parameters represent the square root of the diagonal terms of matrix Φ^r (i.e., the standard deviation of observation errors). The state variables and the parameters obtained from this first-step Kalman filter will be taken as known for the second step.

Once the risk-free term structure is estimated, corporate bonds are grouped into risk classes by their credit rating (AAA, AA, A, or BBB). Given the number of state variables chosen, in this step we have an eight-factor model comprising three factors known from

Table 1. Data description for bond observations used to estimate the risk-free term structure

Bond	Bond type	Range of maturities	Number of bonds	Average trading frequency (percent)
BCU	Bullet	0.7–20 years	10	53
BTU	Bullet	8.7–20 years	4	55
PRC	Amortizing	0–19.6 years	1,411	2

Note: Average trading frequency is calculated as the daily average of number of bonds traded divided by the number of outstanding bonds.

Table 2. Characterization of Chilean inflation-protected corporate bonds with no embedded options, by credit rating: 2003–2006

Credit rating	Average spread (percent)	Standard deviation (percent)	Number of trades	Outstanding number of bonds
AAA	0.70	0.39	728	30
AA	1.25	0.75	2,968	120
A	1.41	1.06	1,541	67
BBB	2.15	1.53	732	29

the first-step estimation, one common factor, and one specific factor for each of the four risk classes used. The sample of risky bonds is ordered by credit rating from the lowest to the highest risk. Therefore, parameters with subscript 4 are the ones governing the process of the common factor, and subscripts 5 to 8 are the AAA, AA, A, and BBB class-specific factors, respectively. Table 4 shows the results of the second step estimation.[10]

Table 4 shows that γ coefficients, which capture the relationship between each risk class and the risk-free term structure, are negative across all ratings, indicating that higher interest rates imply tighter spreads. This result is in line with the findings of Duffee (1998) and Longstaff and Schwartz (1995) for the U.S. market.

Parameters κ represent the mean reversion rate of the state variables. Note that each risk class–specific factor has a mean reversion rate more than ten times larger than the rate of the common factor. This is consistent with the common factor's capturing the long-term relationship among risk classes (i.e., the general level of spreads) and the risk class–specific factor's adjusting the differences between classes at short-term maturities. Lower volatilities for common factors suggest that individual factors capture temporary imbalances, which are very volatile.

Standard deviations of the correlation factors ρ show that in general they are not significant at standard confidence levels. However, this is not surprising since we have included explicit parameters in the framework to capture the two most important correlations hypothesized in the model: among risky and risk-free term structures (through parameters γ) and among risk classes (through common factor x_4). In fact, the nonsignificance of the correlations among class-specific spread factors implies that the comovements of spreads are captured by the common factor.

Table 3. Estimated parameters for the risk-free term structure using in-sample (2003–2006) Chilean government bonds

Parameter	Estimation	Standard deviation
ϕ_{BCU}	0.001327	0.00004
ϕ_{BTU}	0.001688	0.00004
ϕ_{PRC}	0.002196	0.00006
κ_1	0.005344	0.00002
κ_2	0.42833	0.00433
κ_3	6.960949	0.12174
σ_1	0.006583	0.00017
σ_2	0.021357	0.00142
σ_3	0.12773	0.00425
ρ_{21}	0.34305	0.08565
ρ_{31}	–0.0818	0.08020
ρ_{32}	–0.2527	0.04023
λ_1	–0.0008	0.00003
λ_2	0.00908	0.00253
λ_3	–0.2681	0.02350
δ_r^0	0.0106	0.00497

Figures 1 and 2 show the five zero coupon curves generated by the model, along with the implicit spread curves, for two different dates. All the curves have similar shapes, especially for long maturities, for which there are usually fewer observations. The term structures are calibrated using our estimation methodology, which simultaneously uses all risky bond transactions and provides well-behaved risky term structures. Note that the risky and risk-free term structures are similarly shaped, even though the estimation methodology separates their estimations into two different steps.

Figures 3 to 6 show the time series of zero coupon yield curves and the observed yields of corporate bonds pertaining to each rating, represented as a zero coupon bond[11] for durations around which there are enough observations to see whether the model is a good fit.

Figures 3 to 6 illustrate how the model is able to fit the observed yields in spite of the fact that we use the simplest possible specification of the model (one common factor and one specific factor to model each rating term structure). In addition, it is important to note that even when there are no observed prices for a specific class, joint estimation enables transactions from other risk classes to update all the curves by using the historical correlations to estimate unobserved prices.

The goodness of fit of the model to the observed transactions is indicated by the bias and root mean square error (RMSE) measures. Table 5 summarizes the results for the in-sample (2003 to 2006) and out-of-sample (2007) periods.

On average, the model has a RMSE of thirty-two basis points for the four in-sample years, the BBB rating being the class with the highest estimation errors.

Table 5 also shows the rather surprising result that errors in the out-of-sample period are smaller than those in the in-sample period. Table 6 helps to understand this result by analyzing errors and volatility for each year. It shows that every year, annual volatility

Table 4. Estimated parameters for the risky term structures using in-sample (2003–2006) Chilean corporate bonds

Parameter	Estimation	Standard deviation	Parameter	Estimation	Standard deviation
ϕ_{AAA}	0.00254383	0.0005	ρ_{73}	–0.0096813	0.6946
ϕ_{AA}	0.00284541	0.0001	ρ_{83}	0.00030509	0.3745
ϕ_{A}	0.00277438	0.0001	ρ_{54}	–0.1085955	0.1752
ϕ_{BBB}	0.00190698	0.0000	ρ_{64}	–0.30985569	0.0050
ϕ_{SHORT}	0.01092216	0.0003	ρ_{74}	–0.18374735	0.0622
κ_4	0.24726199	0.0043	ρ_{84}	–0.16820467	0.0335
κ_5	4.11788732	0.0401	ρ_{65}	0.05989033	0.3381
κ_6	2.61541273	0.0048	ρ_{75}	0.00401495	0.1194
κ_7	2.97551708	0.0223	ρ_{85}	0.02606305	0.235
κ_8	2.94337642	0.0800	ρ_{76}	0.00524886	0.1000
σ_4	0.01881462	0.0003	ρ_{86}	–0.00542866	0.0865
σ_5	0.33152723	0.0208	ρ_{87}	–0.00268305	0.2060
σ_6	0.43487769	0.0048	λ_4	–0.00135251	0.0000
σ_7	0.33622219	0.0031	λ_5	0.07399285	0.0013
σ_8	0.36717598	0.0090	λ_6	0.00326244	0.0008
ρ_{41}	0.02380569	0.1738	λ_7	–0.06822559	0.0014
ρ_{51}	0.26259078	0.1880	λ_8	–0.05688807	0.0108
ρ_{61}	0.22083723	0.0493	γ_{AAA}	–0.12591521	0.0206
ρ_{71}	0.25417274	0.2394	$\delta_{AAA}{}^{0}$	0.03207525	0.0007
ρ_{81}	–0.04303242	0.1071	γ_{AA}	–0.16288219	0.0092
ρ_{42}	–0.98649389	0.0891	$\delta_{AA}{}^{C}$	1.16747706	0.0266
ρ_{52}	–0.2549835	1.7343	$\delta_{AA}{}^{0}$	0.03249442	0.0002
ρ_{62}	0.15141641	0.3293	γ_{A}	–0.35901622	0.0098
ρ_{72}	0.16261586	0.3124	$\delta_{A}{}^{C}$	1.03211147	0.0608
ρ_{82}	–0.02596767	0.2667	$\delta_{A}{}^{0}$	0.01030767	0.0006
ρ_{43}	0.24538857	0.2709	γ_{BBB}	–0.11069008	0.0171
ρ_{53}	–0.8044074	3.2784	$\delta_{BBB}{}^{C}$	2.00977049	0.0868
ρ_{63}	0.28830856	0.6524	$\delta_{BBB}{}^{0}$	0.01460092	0.0056

of the risky term structures decreased. As expected, the RMSE adjusted by each year's volatility is higher for the out-of-sample period than for the in-sample period.

Driessen (2005) reports an average RMSE of 12.89 basis points (b.p.) using a sample of U.S. corporate bonds from 1991 to 2000. The difference between that result and our in-sample average error is explained mainly by the levels at which the data sets are grouped and market conditions affect the estimations. While Driessen runs his model at the firm level, we group our data into risk classes. Out of 161 available, his sample contains 101 that have at least 2 corporate bonds with trading information for at least 100 weeks (i.e., 21 percent of the weeks in the sample) whereas, if we were to run the same filter in our data set, we would be left with only eight of sixty-five firms in the sample. At the same time, the extreme lack of liquidity induces a higher volatility for our emerging market. In fact, as volatility decreases, we observe a decreasing RMSE: the results for 2007 show an average RMSE of 13 b.p., which is almost the same as the result reported by Driessen (2005).

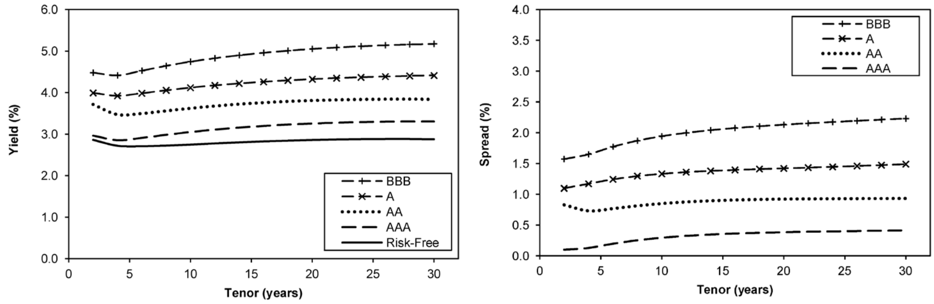

Figure 1. Yield and spread curves estimated with the two-step method for December 6, 2006

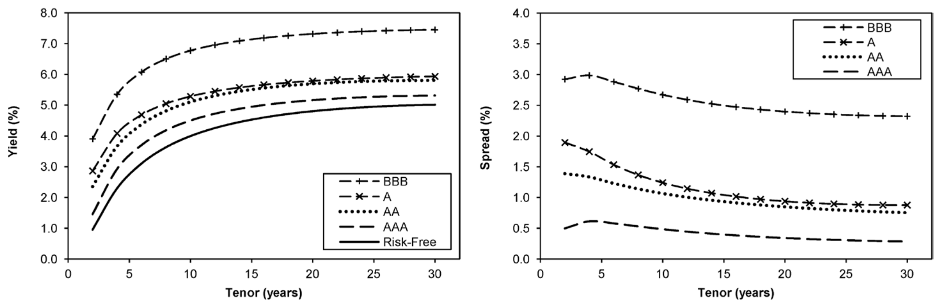

Figure 2. Yield and spread curves estimated with the two-step method for July 2, 2004

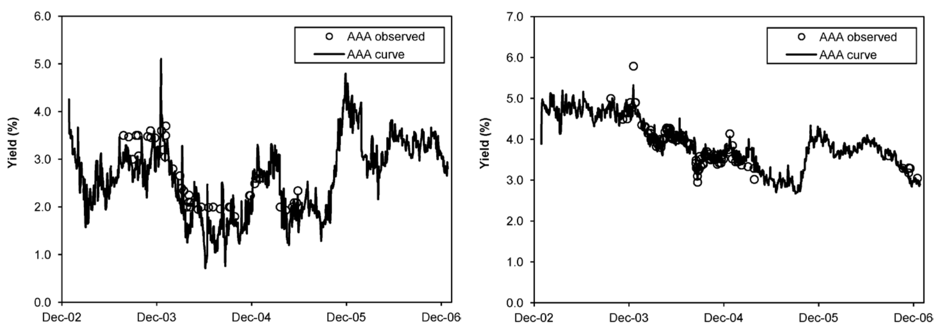

Figure 3. Time series of AAA yield term structure from January 2003 to December 2006 and AAA observed corporate bonds with durations of 2 years (on the left) and 7 years (on the right)

Another important characteristic of a proper dynamic model is its ability to replicate the movements of the interest rate term structure. This is particularly important in a thin market, when information is frequently missing at certain maturities: the prediction step of the filter must be consistent with the historical variances and correlations to increase the reliability of the term structure generated by the model.

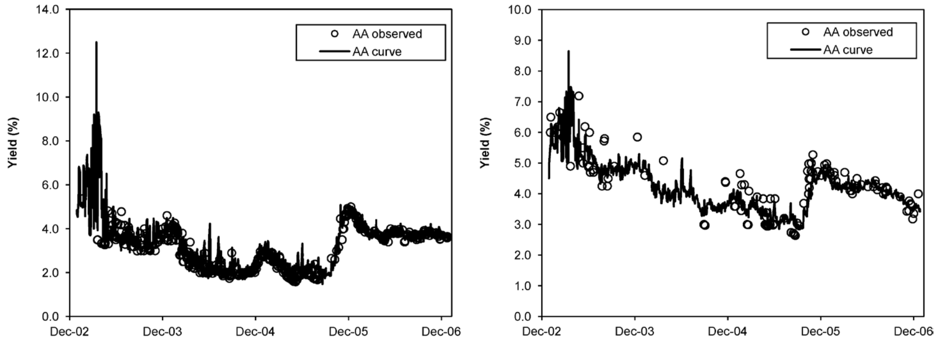

Figure 4. Time series of AA yield term structures from January 2003 to December 2006 and AA observed corporate bonds with durations of 2 years (on the left) and 5 years (on the right)

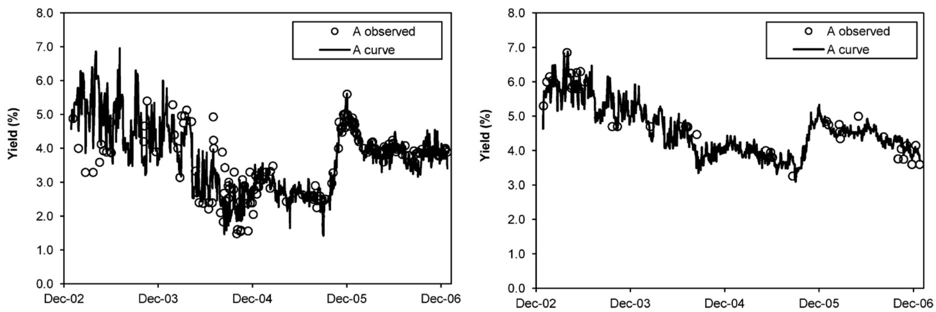

Figure 5. Time series of A yield term structures from January 2003 to December 2006 and A observed corporate bonds with durations of 2 years (on the left) and 5 years (on the right)

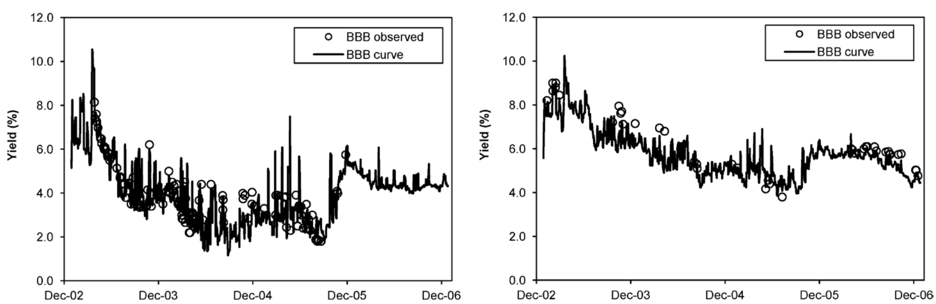

Figure 6. Time series of BBB yield term structures from January 2003 to December 2006 and BBB observed corporate bonds with durations of 2 years (on the left) and 5 years (on the right)

To compare the model's volatility term structure with the empirical volatility of observed bonds, we compute the standard deviation of daily changes in yield, grouping the observations of each rating into duration buckets. Figure 7 shows the comparison,

Table 5. RMSE and bias of the model for in-sample (2003 to 2006) and out–of-sample (2007) periods, measured in basis points

	In sample		Out of sample	
Credit rating	RMSE	Bias	RMSE	Bias
AAA	15	1	12	0
AA	33	1	13	–2
A	32	1	17	5
BBB	41	5	7	2
Total	32	1	13	0

Table 6. RMSE and volatility analysis by year

	2003	2004	2005	2006	2007
RMSE (basis points)	49	36	28	18	13
Annual volatility (σ percent)	1.25	0.68	0.56	0.48	0.26
RMSE/σ	39	52	49	38	51

Note: 2003 to 2006 correspond to the in-sample period, and 2007 is the out-of-sample period.

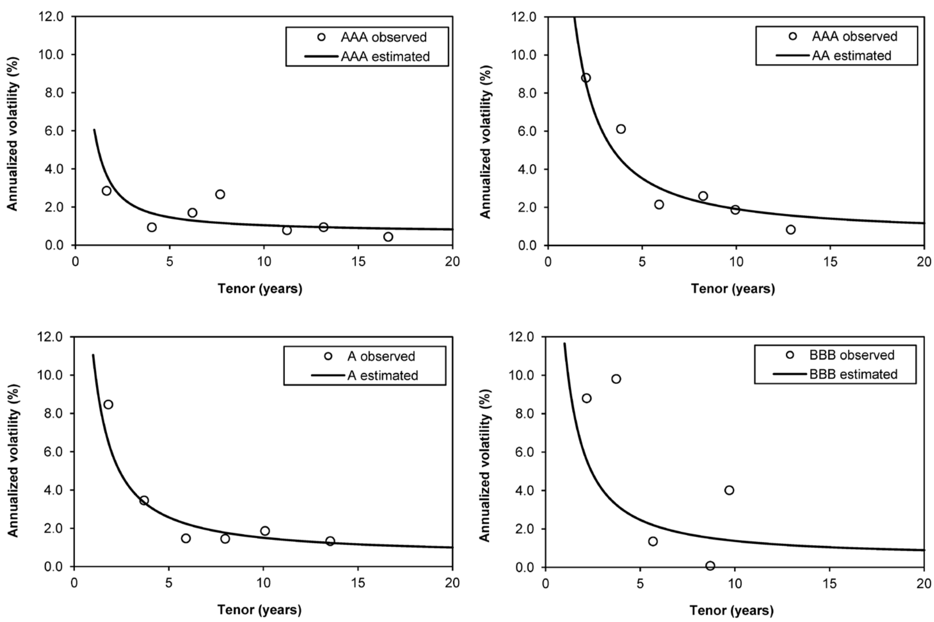

Figure 7. Observed volatilities during the in-sample period between January 2003 to December 2006, compared with the model volatility term structures for each rating

for each risk class, of the annualized observed volatility for the in-sample period with the model's volatility term structure.

Note that model volatilities closely fit empirical volatilities for each rating except BBB. This is consistent with the higher errors for the BBB class, suggesting that this is a more heterogeneous group.

Conclusion

In this paper, we propose a methodology to estimate the credit spread of risky assets in illiquid markets. Our model involves grouping the risky asset observations into risk classes and then estimating a family of term structures, one for the risk-free rate and one for each risky class, based on a dynamic state space model in which each factor follows a Vasicek process. To estimate the model, we propose the use of an extended Kalman filter and a variation of a joint estimation of risky and risk-free term structures. This variation splits the estimation into two steps: the first one for the risk-free term structure and the second for several risky term structures estimated jointly. The methodology makes it possible to take into account notable liquidity differences between these markets.

To test the model, we apply it to a sample of Chilean government and corporate bonds. The results show that the estimation methodology generates consistent spread term structures across all maturities that fit observed risky bond transactions well, even though the model is implemented using the most parsimonious specification.

Finally, joint estimation among the risk classes permits the generation of stable relations between term structures with extremely few observations, using information from all risk classes to update the estimations of the others based on the historical correlations among them.

Notes

1. For a broader description of credit risk models see Bielecki and Rutkowski (2002), Duffie and Singleton (2003), and Lando (2004).

2. The Kalman filter, introduced by Kalman (1960), has been widely used in term structure estimation. See, for example, Chen and Scott (1993), De Jong and Santa-Clara (1999), Duan and Simonato (1999), and Geyer and Pichler (1999). Fontana and Runggaldier (2010) propose an alternative methodology that combines Kalman filter estimations with expectation-maximization techniques to provide a more statistical type of parameter estimation, as opposed to calibration.

3. More general specifications that preserve the affine dynamics of x_t^r may be found in Cheridito et al. (2007).

4. Duffee (1998) and Longstaff and Schwartz (1995), among others, state the negative correlation between corporate spreads and interest rates.

5. Chen and Scott (2003) provide a detailed explanation of the application of the Kalman filter to multifactor CIR models and how to deal with nonnegativity restrictions and state dependence of factor variance. By applying a quasi-linear Kalman filter, they find that even though individual estimation of parameters representing mean reversion rates, long-run averages, and market prices of risk may be biased, their combinations, which govern risk-neutral distribution of state variables, do not show significant biases.

6. We assume the simplest specification for the market price of risk (constant). Duffee and Stanton (2004) find that maximum likelihood estimations may lead to biased estimations of model parameters when more complex definitions of this parameter are proposed.

7. This estimation method has already been applied in interest rate modeling. Ahn (2004) and Mosburger and Schneider (2005) use it to jointly model the interest rates and the exchange rate using local and foreign bonds. In the field of credit spreads, Houweling et al. (2001) postulate a joint estimation of risk-free bonds and corporate spreads using splines, without relating their dynamics.

8. Inflation-protected bonds are denominated in UF *(Unidad de Fomento),* a unit that is updated daily using the change in the previous month Chilean consumer price index.

9. Cortazar et al. (2007) group the observations into maturity buckets and estimate different standard deviations of measurement errors for each bucket. We made a slight modification to this specification and consider that each family of bonds used for calibration has its own standard deviation of measurement errors that remains constant across all maturities.

10. Because we are using only observations with more than one year to maturity and in order to maintain the curves' stability, we include as a short-term observation the previous-day spread of the respective curve for each of the risky term structures. This additional point helps to maintain the relative order among curves, without affecting their ability to fit the observed data.

11. The analysis is done using the duration of the bond as a measure of the maturity of an equivalent, zero coupon bond, because Chilean corporate bonds have heterogeneous coupon payments, which makes it impossible to estimate an internal rate of return curve valid for all bonds.

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