EMPIRICAL PERFORMANCE OF COMMODITY PRICING MODELS: WHEN IS IT WORTHWHILE TO USE A STOCHASTIC VOLATILITY SPECIFICATION?

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We compare the empirical pricing performance of three models: a constant volatility model, a two-factor stochastic volatility model, and a one-factor stochastic volatility model with a model-free implied variance specification. Results of applying these models to oil, copper, and gold derivatives are consistent for all commodities and highlight the relative benefits of the different models implying that in choosing the best model to implement in a real situation, careful consideration must be given to the tradeoffs between effort and precision. We believe our results are not only new, but also relevant for practitioners. © 2015 Wiley Periodicals, Inc. Jrl Fut Mark

1. INTRODUCTION

Commodity derivative markets have grown at an incredible rate in the past decades. According to the Bank for International Settlements (BIS), the notional value of outstanding contracts in June 2013, for OTC commodity derivatives, was USD 2.46 trillion, more than 3.6 times larger than in June 2001. The nature and distribution of the commodity-contingent claims have also changed and options contracts now account for nearly 36% of total notional value.

On the other hand, research on commodity-linked derivatives has also increased, both in quantity and in sophistication. Starting from simple one-factor mean-reverting, constant-volatility models for pricing futures contracts (Vasicek, 1977), the literature has evolved to include the pricing of more complex derivatives, such as options. Although for pricing futures modeling the drift of the risk-neutral process is the most important issue, for pricing options a well-specified dynamics for the volatility becomes crucial.

Despite the empirical evidence of heteroscedasticity in commodity prices (Duffie & Gray, 1995; Litzenberger & Rabinowitz, 1995), it is still common to find in the literature models with several risk factors, but with a constant volatility specification (Cortazar, Milla,

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& Severino, 2008; Cortazar & Eterovic, 2010; Cortazar & Schwartz, 2003; Hilliard & Reis, 1998; Schwartz, 1997; Schwartz & Smith, 2000). In general, these models have several desirable properties, including closed form solutions for most derivatives and good futures pricing. However, little attention has been put on their performance for options pricing.

Although assuming volatility to be constant in multi-factor models may have little implication in terms of goodness-of-fit for futures pricing, it is extremely relevant for options pricing. As the seminal work of Heston (1993), several models that include stochastic volatility have been proposed (Nielsen & Schwartz, 2004; Richter & Sørensen, 2002; Trolle & Schwartz, 2009). The strength of these models is their ability to replicate the time varying volatility behavior of many commodities, obtaining a much better pricing of volatility-sensitive derivatives, like options.

One of the main drawbacks of stochastic volatility models is their implementation complexity because there are no closed-form solutions for derivative prices and intensive numerical methods must be used. Given this difficulty, in order to obtain results in a reasonable amount of time, most stochastic volatility models are implemented restricting the number of risk factors, which could take a toll in the model performance for pricing some derivatives. Recently, there has been a new development in stochastic volatility models that addresses this issue by using the model-free specification of implied variance defined by Britten-Jones and Neuberger (2000) and Jiang and Tian (2005) instead of option price formulas, making them less computationally intensive (Chiang, Hughen, & Saghi, 2015).

We are interested in comparing the performance of these three kinds of models on several grounds. Up to now most papers on the valuation of commodity-contingent claims have been mainly focused on testing the statistical significance of a model when applied to a particular commodity, most commonly, crude oil. Although stochastic volatility models have shown to be statistically significant and consistent with the empirical evidence (Duffie & Gray, 1995), it is not clear to what extent they "perform better" than their constant volatility counterparts from a practitioner's point of view, when implementation issues are taken into consideration. This is why it is critical to understand the magnitude and distribution of pricing errors and the effort required in implementing these kinds of models for different commodities and contracts in order to be able to have a clear picture about the tradeoffs between them.

In this article, we compare the futures and options pricing performance of constant and stochastic volatility models (with both implementations) for several commodities. In order to do this, we use the N-factor Gaussian model developed by Cortazar and Naranjo (2006) to represent a constant volatility framework, the Trolle and Schwartz (2009) for the traditional two-factor stochastic volatility approach and the Chiang et al. (2015) for the recent one-factor stochastic volatility model with a model-free implied variance specification. We choose Cortazar and Naranjo (2006) because its canonical representation for N-factor Gaussian models nests several of the existing Gaussian models in the literature as special cases (Brennan & Schwartz, 1985; Cortazar & Schwartz, 2003; Gibson & Schwartz, 1990; Schwartz, 1997; Schwartz & Smith, 2000). On the other hand, the Trolle and Schwartz (2009) specification is selected because, having already been applied to oil, it has the added flexibility to allow for volatility components not to be fully spanned by the spot (futures) market in what has been called "unspanned stochastic volatility" factors (Collin-Dufresne & Goldstein, 2002). Using this specification will allow us to check if this characteristic, which would imply that options are not redundant securities, may be present not only for oil, but also eventually for other commodities. Finally, we use the novel Chiang et al. (2015) approach as an "in between" model because it is a stochastic volatility model, but with a simpler estimation procedure.

Following Schwartz (1997), we analyze three different commodities: oil, copper, and gold. We do not only focus on model results (statistical significance and futures and options pricing errors), but we also consider implementation issues in order to analyze the strengths of each model specification. Execution times are measured as a proxy of implementation complexity for each model. This quantitative analysis may help, from the practitioner's perspective, to balance the tradeoffs between a better fit and an easier implementation.

Empirical performance of alternative commodity models that do not restrict themselves to statistical tests are limited. In a seminal work, Schwartz (1997) analyzes futures pricing performance of three different models each one applied to crude oil, high-grade copper, and gold. Also, Hughen (2010) compares, in terms of options and futures pricing fit, a stochastic volatility model with an affine constant volatility one. However the paper only analyzes the 3-factor model specification, does not use options prices in the calibration process, restricts the analysis only to oil and does not analyze cross-sectional performance differences. In an analysis for interest rate derivatives Bakshi, Cao, and Chen (1997) examine the term structure option pricing differences between the Black–Scholes model and several other specifications that include and combine stochastic interest rates, stochastic volatility and random jumps diffusion.

We use oil, copper and gold futures and options settlement data of the New York Mercantile Exchange (NYMEX) from January 2006 to May 2013. The Kalman filter, in its traditional and extended versions, is used together with maximum likelihood, as the estimation procedure.

To our knowledge this paper is the first to compare stochastic volatility models against their constant volatility benchmarks using oil, copper and gold options and futures prices. It is the first to calibrate the Chiang et al. (2015) for the three commodities and also extends Trolle and Schwartz (2009), by expanding dates and commodities, and Cortazar and Naranjo (2006), by including options, and not only futures, in the analysis.

This paper is structured as follows. Section 2 describes the benchmark models. Section 3 describes the crude oil, copper and gold data. Section 4 analyzes and discusses the empirical results. Section 5 concludes.

2. MODELS

In this section the three benchmark models used to compare the empirical performance of the constant versus the stochastic volatility models are described.

2.1. Stochastic Volatility Model: Trolle and Schwartz (2009) (TS)

We now briefly describe the Trolle and Schwartz (2009), TS model which, besides its ability to account for stochastic volatility, offers a tractable framework to price commodity derivatives in the presence of unspanned stochastic volatility. It has the flexibility to model a market where options are not redundant securities (Collin-Dufresne & Goldstein, 2002).

The TS model is chosen, among other reasons, because it is based on the Heath, Jarrow and Morton (1992), HJM model, which has the advantage, over the typical affine models, of making it easier to include the parameter restrictions on volatility to be unspanned.[1] In its most general form, the model has five risk factors, three for the futures prices and two for the volatility.

[1]See Cortazar and Schwartz (1994), Miltersen and Schwartz (1998), and Miltersen (2003) for other HJM-type models for pricing commodity derivatives.

Following Cortazar and Schwartz (1994), a process for the spot price and the forward cost- of-carry is specified. Let $S(t)$ denote the spot price of the commodity at time t and $\delta(t)$ the spot cost-of-carry. Let $y(t,T)$ denote the t-time instantaneous cost-of-carry curve at time T ($\delta(t) = y(t,t)$). To account for stochastic volatility let $v_1(t)$ and $v_2(t)$ be two volatility factors affecting $S(t)$ and $y(t,T)$.

The processes for $S(t)$, $y(t,T)$, $v_1(t)$, and $v_2(t)$, under the risk-neutral measure, are:

$$\frac{dS(t)}{S(t)} = \delta(t)dt + \sigma_{S1}\sqrt{v_1(t)}dW_1^Q(t) + \sigma_{S2}\sqrt{v_2(t)}dW_2^Q(t) \tag{1}$$

$$dy(t,T) = \mu_y(t,T)dt + \sigma_{y1}(t,T)\sqrt{v_1(t)}dW_3^Q(t) + \sigma_{y2}(t,T)\sqrt{v_2(t)}dW_4^Q(t) \tag{2}$$

$$dv_1(t) = (\eta_1 - \kappa_1 v_1(t) - \kappa_{12}v_2(t))dt + \sigma_{v1}\sqrt{v_1(t)}dW_5^Q(t) \tag{3}$$

$$dv_2(t) = (\eta_2 - \kappa_{21} v_1(t) - \kappa_2 v_2(t))dt + \sigma_{v2}\sqrt{v_2(t)}dW_6^Q(t) \tag{4}$$

$$(dW(t))(dW(t))' = \begin{pmatrix} 1 & 0 & \rho_{13} & 0 & \rho_{15} & 0 \\ 0 & 1 & 0 & \rho_{24} & 0 & \rho_{26} \\ \rho_{13} & 0 & 1 & 0 & \rho_{35} & 0 \\ 0 & \rho_{24} & 0 & 1 & 0 & \rho_{46} \\ \rho_{15} & 0 & \rho_{35} & 0 & 1 & 0 \\ 0 & \rho_{26} & 0 & \rho_{46} & 0 & 1 \end{pmatrix}$$

where $dW_i^Q(t)$ corresponds to a Wiener process under the risk-neutral measure.

The futures price of a contract expiring at time T, $F(t,T)$, is:

$$F(t,T) = S(t)\exp\left\{\int_t^T y(t,u)du\right\} \tag{5}$$

Under no-arbitrage conditions the drift of the instantaneous futures return should be zero, thus:

$$\begin{aligned} \frac{dF(t,T)}{F(t,T)} = {} & \sqrt{v_1(t)}\left(\sigma_{S1}dW_1^Q(t) + \int_t^T \sigma_{y1}(t,u)du\ dW_3^Q(t)\right) \\ & + \sqrt{v_2(t)}\left(\sigma_{S2}dW_2^Q(t) + \int_t^T \sigma_{y2}(t,u)du\ dW_4^Q(t)\right) \end{aligned} \tag{6}$$

The ability of the TS model to account for unspanned stochastic volatility can be seen in Equation (6) where volatility of futures prices is shown to depend on $v_1(t)$ and $v_2(t)$ but not on $dW_5^Q(t)$ and $dW_6^Q(t)$. Therefore if $dW_5^Q(t)$ and $dW_6^Q(t)$ have a low correlation with the risk processes of the spot and forward cost-of-carry curve ($dW_i^Q(t)$, $i = 1, \ldots, 4$) then options (which are highly sensitive to volatilities) cannot be hedged using only futures.

In order to estimate the model it is necessary to specify the drift and instantaneous volatility of the forward cost of carry curve. A drift condition, analogous to the one developed by Heath, Jarrow, and Morton (1992) in forward rate term structure models, can be obtained and the volatility is chosen in a way that the forward cost of carry curve can be expressed as a linear function of a finite number of state variables.[2] Then it follows

$$\mu_y(t,T) = -\left(\nu_1(t)\sigma_{y1}(t,T)\left(\rho_{13}\sigma_{S1} + \int_t^T \sigma_{y1}(t,u)du\right) + \nu_2(t)\sigma_{y2}(t,T)\left(\rho_{24}\sigma_{S2} + \int_t^T \sigma_{y2}(t,u)du\right)\right) \tag{7}$$

where

$$\sigma_{yi}(t,T) = \alpha_i e^{-\gamma_i(T-t)} \tag{8}$$

Under such conditions $F(t,T)$ is given by

$$F(t,T) = S(t)\frac{F(0,T)}{F(0,t)}\exp\left\{\sum_{i=1}^{2}\left(x_i(t)\frac{\alpha_i}{\gamma_i}\left(1-e^{-\gamma_i(T-t)}\right) + \phi_i(t)\frac{\alpha_i}{2\gamma_i}\left(1-e^{-2\gamma_i(T-t)}\right)\right)\right\} \tag{9}$$

where $x_i(t)$ and $\phi_i(t)$, $i = 1,2$, are the resulting state variables from the transformation of the HJM-type model to an affine Markovian specification. They follow a process described in Appendix A.1.

Defining the new state variable $s(t) = \log S(t)$, log futures prices follow an affine function of the state variables:

$$\begin{aligned}\log(F(t,T)) &= logF(0,T) - logF(0,t) \\ &\quad + s(t) + \sum_{i=1}^{2}\left(x_i(t)\frac{\alpha_i}{\gamma_i}\left(1-e^{-\gamma_i(T-t)}\right) + \phi_i(t)\frac{\alpha_i}{2\gamma_i}\left(1-e^{-2\gamma_i(T-t)}\right)\right)\end{aligned} \tag{10}$$

where the dynamics of $s(t)$ is given by

$$\begin{aligned}ds(t) = &\left(y(0,t) + \sum_{i=1}^{2}\alpha_i(x_i(t)+\phi_i(t)) - \frac{1}{2}\left(\sigma_{S1}^2\nu_1(t) + \sigma_{S2}^2\nu_2(t)\right)\right)dt \\ &+ \sigma_{S1}\sqrt{\nu_1(t)}dW_1^Q(t) + \sigma_{S2}\sqrt{\nu_2(t)}dW_2^Q(t)\end{aligned} \tag{11}$$

Based on Heston (1993) European options on futures are priced applying the Fourier inversion theorem. A similar approach has been used by Collin-Dufresne and Goldstein (2003) and Richter and Sørensen (2002), among others. Letting K be the strike price and T_0 the option expiration time on a future contract expiring at T_1 the price of put is given by

$$\mathscr{P}(t,T_0,T_1,K) = P(t,T_0)\left(KG_{0,1}(log(K)) - G_{1,1}(log(K))\right) \tag{12}$$

where $P(t,T_0)$ is the price of a zero-coupon bond with $T_0 - t$ maturity and $G_{a,b}(y)$ is defined as

$$G_{a,b}(y) = \frac{\Psi(a,t,T_0,T_1)}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im\left(\Psi(a+iub,t,T_0,T_1)\exp^{(-iuy)}\right)}{u}du \tag{13}$$

[2]Bhar and Chiarella (1997) present the conditions under which HJM-type models are Markovian.

Here $\Psi(u,t,T_0,T_1) = E_t^Q\left[e^{ulog(F(T_0,T_1))}\right]$ represents the transform of $F\ (T_0,T_1)$ and has an affine representation given by the solution of a non-trivial system of ordinary differential equations with no closed-form solution (See Appendix A.2).

For calibration purposes the dynamics of the commodity has to be stated under the actual probability measure. In order to do this a market price of risk is specified. Based on an affine formulation widely used in the literature, the market price of risk is defined as:

$$\begin{aligned}\Lambda_i(t) &= \lambda_i\sqrt{\nu_1(t)},\ i = 1,3,5\\ \Lambda_i(t) &= \lambda_i\sqrt{\nu_2(t)},\ i = 2,4,6\end{aligned} \tag{14}$$

Then, the processes under the actual probability measure can be obtained by substituting $dW^Q(t)$ by

$$dW_i^Q(t) = dW_i^P(t) - \Lambda_i(t)dt \tag{15}$$

Roughly speaking in order to implement the TS model for options pricing it is necessary first, to numerically solve for each contract an ordinary differential equations system to get the above mentioned transform, then, also for each contract, to apply numerical integration algorithms twice to finally get the option price. This theoretically complex framework and also numerically sophisticated formulation, has practical consequences that lead to the question of when it is worth to make the effort, as we will see in further sections.[3]

2.2. Constant Volatility Model: Cortazar and Naranjo (2006) (CN)

We now describe the main features of the Cortazar and Naranjo (2006) N-factor Gaussian model for a commodity spot price that will later be used to analyze the empirical performance of a constant volatility approach. This model generalizes 2 and 3-factor models found in the literature (Cortazar & Schwartz, 2003; Gibson & Schwartz, 1990; Schwartz, 1997; Schwartz & Smith, 2000). It provides a framework based on the $A_0(N)$ canonical representation of Dai and Singleton (2000) for the term structure, extending existing commodity pricing models to an arbitrary number of factors. One of the main advantages of this model, compared to the stochastic volatility ones, is its relatively simple implementation and the existence of closed-form analytic formulas for futures and options prices.

Let $S(t)$ be the spot price of the commodity at time t, and μ the long-term growth rate. Then the process for the spot price of the commodity is:

$$logS(t) = 1'X(t) + \mu t \tag{16}$$

where $X(t)$ is the $N \times 1$ vector of unobservable state variables with a process, under the actual probability measure, given by:

$$dX(t) = -AX(t)dt + \Sigma dW(t) \tag{17}$$

[3]The model presented here is time-nonhomogeneous and fits the initial futures curve by construction. In order to estimate the model, the initial forward cost of carry curve is assumed to be flat and equal to a constant ϕ. Also, the model is over identified, therefore $\eta_i, i = 1,2$, is normalized to one, to achieve identification.

where

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_N \end{bmatrix} \text{and } \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N \end{bmatrix}$$

are $N \times N$ matrices with positive entries and $dW(t)$ is the $N \times 1$ vector of correlated Wiener processes such that

$$(dW(t))(dW(t))' = \Omega dt = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{12} & 1 & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1N} & \rho_{2N} & \cdots & 1 \end{bmatrix} dt$$

where $\rho_{ij} \in [-1, 1]$ are the instantaneous correlation between state variables i and j.[4]

Without specifying a stochastic process Σ, the model implies that the state variables follow a multivariate Normal distribution where each variable, except for the first one, reverts to zero at a speed rate a_i.

Futures pricing formulas are easily obtained under the risk-neutral measure by assuming constant risk premiums λ_i:

$$dX(t) = -(\lambda + AX(t))dt + \Sigma dW^Q(t) \tag{18}$$

Using no-arbitrage arguments, the futures price becomes:

$$\begin{aligned} F(X(t), t, T) = \exp\Bigg(& X_1(t) + \sum_{i=2}^{N} e^{-a_i(T-t)} X_i(t) + \mu t + \left(\mu - \lambda_i + \frac{1}{2}\sigma_1^2\right)(T-t) \\ & - \sum_{i=2}^{N} \frac{1 - e^{-a_{i(T-t)}}}{a_i} \lambda_i + \frac{1}{2} \sum_{i,j \neq 1} \sigma_i \sigma_j \rho_{ij} \frac{1 - e^{-(a_i + a_j)(T-t)}}{a_i + a_j} \Bigg) \end{aligned} \tag{19}$$

Following Hilliard and Reis (1998) and Miltersen and Schwartz (1998), a Black–Scholes-type formula can be derived for European futures options under the CN specification. Let σ_F be the instantaneous volatility of the returns on futures, then the price of a European put option at time t expiring at T_0 and with strike price K over a future contract maturing at T_1 is given by

$$\mathscr{P}(t, T_0, T_1, K) = P(t, T_0)(KN(-d_2) - F(t, T_1)N(-d_1)) \tag{20}$$

where

$$d_1 = \frac{log\left(\frac{F(t,T_1)}{K}\right) + \frac{1}{2}\nu^2}{\nu}, \; d_2 = d_1 - \nu \tag{21}$$

[4]It is important to note that α_1 has been fixed at zero in order to have a non-stationary process for the underlying spot price, as it is commonly assumed in the literature.

$N(\cdot)$ is the cumulative standard normal distribution function and $P(t, T_0)$ is the price of a zero-coupon bond at time t, expiring at T_0 and ν is the volatility term.

Then

$$\nu^2 = \int_t^{T_0} \sigma_F(u, T_1)^2 du = \int_t^{T_0} \sum_{i=1}^{N}\sum_{j=1}^{N} \sigma_i \sigma_j \rho_{ij} e^{-(a_i+a_j)(T_1-u)} du \tag{22}$$

where ν^2 is the average of the instantaneous variance of the futures return innovations over the life of the options.

In contrast with the TS model, the CN framework provides closed-form solutions for futures and options pricing formulas, making it easier to apply standard estimation procedures.

2.3. Stochastic Volatility Model with a Model-Free Implied Variance Specification: Chiang et al. (2015) (CHS)

We now describe the main features of the Chiang et al. (2015) four-factor model that will later be compared with the two previous models. This is a stochastic volatility model (similar, in this sense, to the TS model), but with a model-free specification for the implied variance.

The goal of this model is to estimate state variables from observed prices of oil futures, options, and equity. The four-factor affine model is defined by the following risk-neutral process:

$$dX_t = (A^* + B^* X_t)dt + \Sigma_t dW^* \tag{23}$$

where X_t is the four-dimensional state variable of oil factors defined as $X_t = \left(X_{s,t}, X_{l,t}, X_{p,t}, X_{v,t}\right)$ and

$$A^* = \begin{bmatrix} 0 \\ \mu_1 \\ 0 \\ \mu_2 \end{bmatrix}, \; B^* = \begin{bmatrix} -\kappa_1 & 0 & 0 & 0 \\ 0 & -\kappa_2 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -\kappa_3 \end{bmatrix}, \; \Sigma_t \Sigma_t' = \Omega_0 + \Omega_1 X_{v,t}$$

$$\Omega_0 = \begin{bmatrix} \sigma_1^2 & \rho_1\sigma_1\sigma_2 & -\sigma_1^2 & 0 \\ \rho_1\sigma_1\sigma_2 & \sigma_2^2 & -\rho_1\sigma_1\sigma_2 & 0 \\ -\sigma_1^2 & -\rho_1\sigma_1\sigma_2 & \sigma_1^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; \Omega_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \rho_2\sigma_3 \\ 0 & 0 & \rho_2\sigma_3 & \sigma_3^2 \end{bmatrix}$$

The spot price of oil is defined as $logS_t = X_{s,t} + X_{p,t}$.

The instantaneous risk premium associated with this model is:

$$\Lambda_t = A - A^* + (B - B^*)X_t = \Omega_t \lambda \tag{24}$$

where $\Omega_t = dX_t dX_t'$.[5]

The futures price is defined as (see Chiang et al. (2015)):

$$F(t, \tau) = e^{\chi(\tau) + \eta_s(\tau)X_{s,t} + \eta_l(\tau)X_{l,t} + \eta_p(\tau)X_{p,t}} \tag{25}$$

[5]This is the same risk premia specification assumed for the TS model.

The model also extracts information from options prices using a model-free approach. Following Britten-Jones and Neuberger (2000) and Jiang and Tian (2005), the implied variance of an asset between t and T, $IV_{t,T}$, is defined as:

$$(T-t)\times IV_{t,T} = E_t^*\left[\int_t^T\left(\frac{dF_{s,T-s}}{F_{s,T-s}}\right)^2\right] = 2\int_0^\infty \frac{C_t(T-t,K)e^{y_{t,T-t}(T-t)} - \left(F_{t,T-t}-K\right)^+}{K^2}dK \quad (26)$$

where $C_t(\tau,K)$ is the price of a European call option at time t, strike K, and maturity of τ, and $y_{t,\tau}$ is the time-t yield of a risk-free discount bond with maturity τ. It can be shown that this implied variance is linear in the volatility state variable:

$$IV_{t,T} = \gamma(t,T) + \delta(t,T)X_{v,t} \quad (27)$$

Finally, the model uses equity data to extract information about oil state variables. If the model were perfectly specified and the futures and options prices were observed without error, the instantaneous excess returns would be given by:

$$dX_t^e = (dX_t - E_t[dX_t]) + \Lambda_t dt = \Sigma_t dW_t + \Lambda_t dt \quad (28)$$

Then, the model for oil equity returns would be:

$$\frac{dP_t}{P_t} - r_{f,t}dt = b_M\left(\frac{dM_t}{M_t} - r_{f,t}dt\right) + b'_{oil}dX_t^e + dz_t \quad (29)$$

where $\frac{dM_t}{M_t} - r_{f,t}dt$ is the instantaneous excess return on the market portofolio.

The CHS stochastic volatility model simplifies the estimation requirements of the TS model, which required many numerical approximations.

As in Trolle and Schwartz (2009), European options on futures are priced applying the Fourier inversion theorem. Letting K be the strike price and T_0 the option expiration time on a futures contract expiring at T_1 the price of a put is given by:

$$\mathscr{P}(t,T_0,T_1,K) = P(t,T_0)\big(KG_{0,1}(log(K)) - G_{1,1}(log(K))\big) \quad (30)$$

where $P(t,T_0)$ is the price of a zero-coupon bond with $T_0 - t$ maturity and $G_{a,b}(y)$ is defined as

$$G_{a,b}(y) = \frac{\Psi(a,t,T_0,T_1)}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im(\Psi(a+iub,t,T_0,T_1)e^{-iuy})}{u}du \quad (31)$$

Here $\Psi(u,t,T_0,T_1) = E_t^Q\left[e^{ulog(F(T_0,T_1))}\right]$ represents the transform of $F(T_0,T_1)$ and has an affine representation given by the solution of a non-trivial system of ordinary differential equations with no closed-form solution (See Appendix A.3).

3. DATA

This section describes the oil, copper, and gold data that will be used to analyze the empirical performance of the previously described commodity models. The data consist on daily observations of settlement prices, open interest, and volume, for futures and options, between January 2006 and May 2013. For oil, we consider the West Texas Intermediate

TABLE I
Oil Data (From January 2006 to May 2013. Daily Observations)

	Futures			*Options*					
Futures Contract	*Avg. Price*	*Avg. Maturity*	*Avg. Open Interest*	*N° Puts*	*N° Calls*	*Avg. Put Price*	*Avg. Call Price*	*Avg. Put Imp. Vol. (%)*	*Avg. Call Imp. Vol. (%)*
F1	82.31	0.083	287016	8350	8748	1.106	1.142	44.16	40.37
F2	82.973	0.166	132036	9907	10006	1.844	1.928	40.25	37.12
F3	83.481	0.249	84095	9944	9925	2.577	2.738	38.16	35.73
F4	83.884	0.333	66127	9778	9520	3.204	3.463	36.69	34.91
F5	84.205	0.416	53623	9502	8979	3.792	4.061	36.00	34.26
F6	84.461	0.5	46728	8869	8402	4.288	4.617	35.26	33.48
MD1	84.827	0.665	73521	9415	8936	5.194	5.47	34.99	32.61
MD2	85.157	0.915	59311	7824	7177	6.232	6.536	33.47	31.69
D1	85.248	1.546	105709	10002	9609	7.668	8.06	31.72	29.12
D2	84.929	2.547	55105	9809	8695	8.983	9.934	27.73	26.51
D3	84.659	3.548	33066	8830	7850	10.012	11.2	26.45	25.04
D4	84.71	4.549	20473	6136	5270	10.905	12.868	24.92	24.60

Notes: The Futures Contract column: Fi denotes the first i-month contracts; MDi denotes the i-following contract with expiration either in March, June, September, or December; Di denotes the i-following contract with expiration in December. Prices are expressed in US$ and maturities in years. The Avg. Imp. Vol. represents the average Black and Scholes implied volatilities for each maturity.

crude oil data (WTI) from the New York Mercantile Exchange (NYMEX). For copper and gold we use high grade copper (HG) and gold (GC) traded at the Commodity Exchange (COMEX).

Tables I–III describe the data used for oil, copper, and gold, respectively. Following Trolle and Schwartz (2009), liquidity considerations for each commodity are taken into account in building the data sets. Daily observations on contracts prices are selected

TABLE II
Copper Data (From January 2006 to May 2013. Daily Observations)

	Futures			*Options*					
Futures Contract	*Avg. Price*	*Avg. Maturity*	*Avg. Open Interest*	*N° Puts*	*N ° Calls*	*Avg. Put Price*	*Avg. Call Price*	*Avg. Put Imp. Vol. (%)*	*Avg. Call Imp. Vol. (%)*
F1	3.276	0.082	8546	1246	1363	0.027	0.028	43.86	43.55
F2	3.276	0.165	29621	5772	6035	0.046	0.05	40.59	39.02
F3	3.277	0.249	29956	5729	5499	0.067	0.081	38.91	37.70
F4	3.278	0.332	17979	4205	4069	0.09	0.113	38.40	37.61
F5	3.277	0.416	9389	3092	2898	0.107	0.139	38.36	36.41
F6	3.275	0.499	5718	2096	2017	0.129	0.163	37.74	36.30
MD1	3.267	0.646	6520	2426	2100	0.17	0.219	37.78	37.99
MD2	3.253	0.848	2623	769	444	0.202	0.232	35.39	34.72

Notes: The Futures Contract column: Fi denotes the first i-month contracts; MDi denotes the i-following contract with expiration either in March, May, September or December. Prices are expressed in US$ and maturities in years. The Avg. Imp. Vol. represents the average Black and Scholes implied volatilities for each maturity.

TABLE III
Gold Data (From January 2006 to May 2013. Daily Observations)

	Futures			*Options*					
Futures Contract	*Avg. Price*	*Avg. Maturity*	*Avg. Open Interest*	*N° Puts*	*N ° Calls*	*Avg. Put Price*	*Avg. Call Price*	*Avg. Put Imp. Vol. (%)*	*Avg. Call Imp. Vol. (%)*
F1	1111.278	0.08	148564	4608	5152	5.117	5.671	25.41	25.65
F2	1114.707	0.241	138392	9631	10129	11.519	14.681	24.42	25.17
F3	1117.883	0.408	46464	9572	9997	20.9	27.159	23.99	25.28
F4	1120.963	0.575	21960	8874	9360	30.096	39.582	23.98	25.35
F5	1124.037	0.742	16560	7721	8519	40.216	51.692	24.14	25.54
F6	1127.172	0.908	12790	6090	6780	50.507	63.131	24.59	25.60
JD1	1141.244	1.578	10112	3655	4708	83.081	104.012	24.81	25.62
JD2	1153.732	2.078	8160	1235	1924	129.536	149.678	26.56	26.75
JD3	1168.12	2.579	6441	444	760	134.755	176.939	25.62	26.04
JD4	1183.991	3.079	4164	295	66	128.333	187.092	26.98	25.25
JD5	1201.505	3.579	4427	254	137	154.651	253.023	32.18	26.22

Notes: The Futures Contract column: Fi denotes the first i-month contracts; JDi denotes the first i contracts with expiration either in June or December after a year from date. Prices are expressed in US$ and maturity in years. The Avg. Imp. Vol. represents the average Black and Scholes implied volatilities for each maturity.

according to the level of open interest and specific liquidity patterns for each commodity. This procedure leaves twelve generic futures contracts for oil: the first 6-month contracts (F1–F6), the following two contracts with expiration either in March, June, September, and December (MD1–MD2), and the next four contracts with expiration in December (D1–D4). For copper and gold, the selection process leaves eight and eleven generic futures contracts, respectively: the first 6-month contracts (F1–F6) and the following two first contracts with expiration either in March, May, September, and December (MD1–MD2) for copper, and the first 6-month contracts (F1–F6) and the first five contracts with expiration either in June and December (JD1–JD5) after a year, for gold.

Given that the commodity markets trade only American options and that for simplicity they are priced using European options formulas, only at-the-money and out-of-the-money options are considered to reduce the size of the early exercise premium. The options are classified in eleven moneyness intervals, ranging from 0.78 to 1.22 years, and the closest contract to the mean of each interval is selected.

Figure 1 shows the spot price evolution of the nearest contract for the three commodities. It is important to note the impact of the financial crisis of 2008 on the spot prices of the three commodities. Figure 2 shows that this impact is not only on the price levels, but also on the Black–Scholes implied volatility of the closest-to-maturity contract.

Option formulas used to compare the three models assume that interest rates are uncorrelated with futures prices. This assumption holds because the correlations between interest rates and futures returns are very low for all three commodities.[6]

In the following section, results of the three models for each of the three commodities are presented. In order to test the empirical performance of the models, the period from January 2012 to May 2013 is defined as the out-of-data set. Also to isolate the effect of the financial crisis data, sub-samples are constructed: Panel A represents the full in-sample

[6]For example, the correlation between oil future returns and 6-month interest rates is 0.019 for the shortest contract and 0.018 for the longest contract.

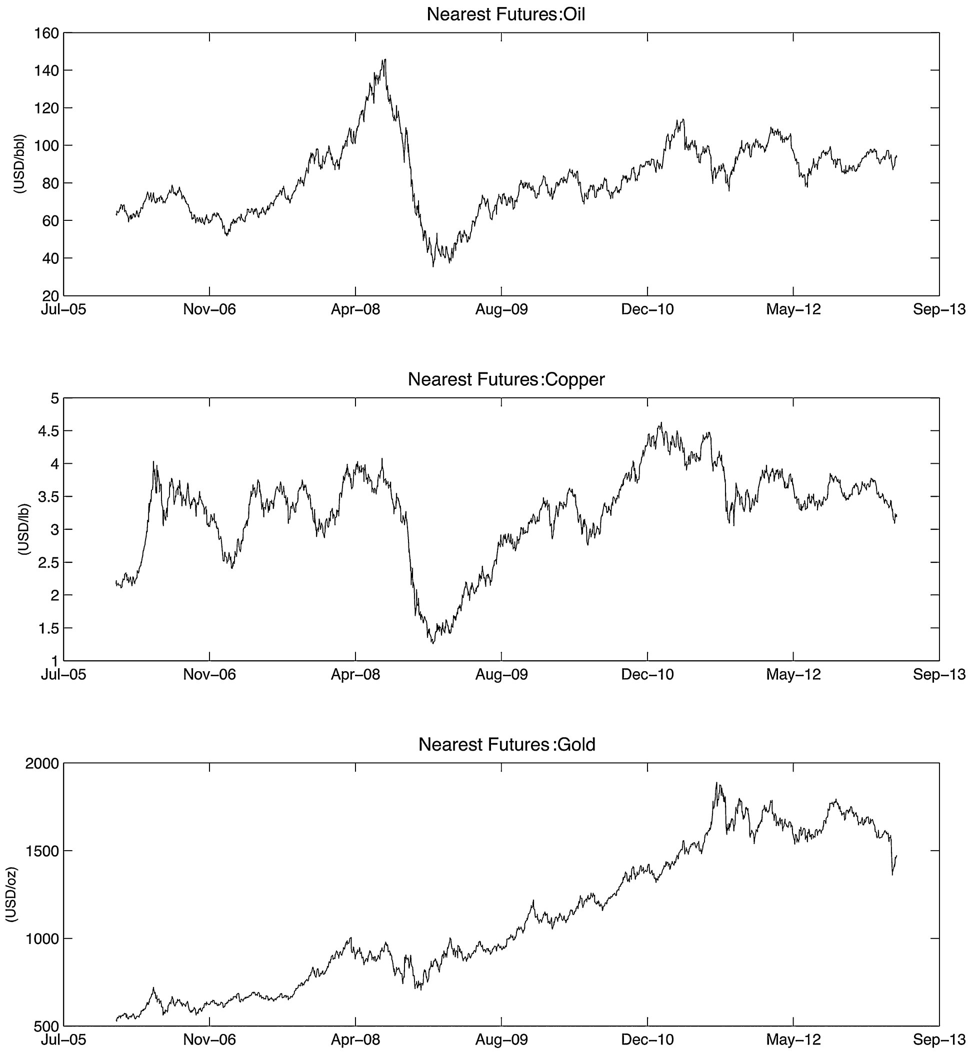

FIGURE 1
Spot price of Oil, Copper, and Gold using the closest-to-maturity futures (nearest futures) between January 2006 and May 2013.

period, from January 2006 to December 2011, Panel B from January 2006 to December 2007, Panel C from January 2008 to December 2009, and Panel D from January 2010 to December 2011. Panel E represents the out-of-sample period.

4. RESULTS

In this section, we present the results of applying the three models to each of the panels of data for every commodity. We start by analyzing a subset of the parameter values for each model. Next we compare the pricing performance of each model. Finally we present a measure of the implementation complexity which adds valuable information when choosing the model to use.

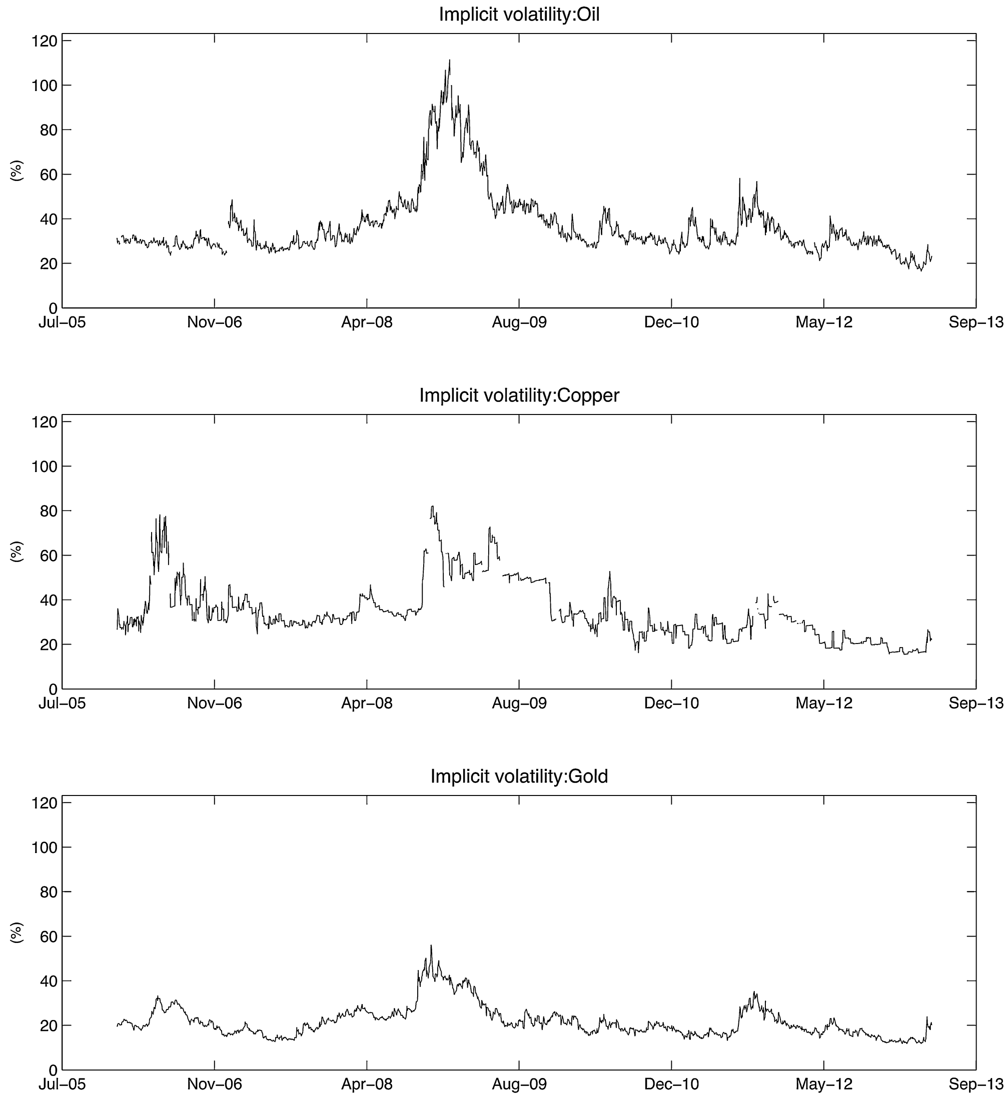

FIGURE 2

Lognormal Implied Volatility using Black–Scholes models for the closest-to-maturity option contract between January 2006 and May 2013.

4.1. Estimation Results

This section discusses some results of the Kalman filter and maximum likelihood estimation of each model. To make the models comparable among each other, five, four, and three-factor specifications for the CN model are estimated.[7] Appendix B explains the estimation procedure.

In the TS model for oil, our parameter values are consistent with those in Trolle and Schwartz (2009). The only parameters not statistically significant are the risk premiums, λ_i, which is common for this type of models (Cortazar, Kovacevic, & Schwartz, 2015), but does

[7]The parameter values of all model specifications are not reported, but are available from the authors upon request.

not affect the pricing of futures and options. Volatility factors, $\nu_1(t)$ and $\nu_2(t)$, exhibit low correlation with the spot price ($\rho_{15} = -0.2109$, $\rho_{26} = -0.3087$) and forward cost-of-carry curve ($\rho_{35} = 0.1095$, $\rho_{46} = -0.0282$), which strongly suggests the presence of unspanned stochastic volatility. The estimates of the mean reverting coefficients, $\kappa_1 = 6.8588$, $\kappa_{12} = -5.1711$, $\kappa_{21} = -0.1710$, and $\kappa_2 = 0.8214$, of the volatility components indicate that most of the transitory shocks to the volatility are absorbed by $\nu_1(t)$, which strongly reverts to $\nu_2(t)$. As expected, observation standard errors, $\sigma_{fut} = 0.0042$ and $\sigma_{op} = 0.0182$ are highly significant and relatively low.

In the CN model for oil, all mean reversion parameters, a_i, as well as the volatility ones, σ_i, are statistically significant and show strong mean reversion. Risk premiums and the long-term growth rate parameters are not significant for most panels, which is consistent with Schwartz (1997). The standard deviation observation errors are small, but highly significant.

In the CHS model for oil, parameters are statistically significant across, except for risk premium parameters. The standard deviation of observation errors in this model is similar to the value observed for the TS model. Also, the standard deviation observation error for options (σ_{op}) is smaller those for the CN model.

The three models applied to copper yield similar results than those reported for oil. For all models, most parameters are statistically significant, with the exception of the risk premiums.

In the results of the TS model for copper,[8] it is interesting to note that correlations between the volatility components and the other variables are quite low for all panels ($\rho_{35} = 0.0088$), which is consistent with the existence of unspanned stochastic volatility. Also both volatility components present relatively high mean reversion coefficients, $\kappa_1 = 10.8751$ and $\kappa_2 = 1.6236$, compared to oil estimates, $\nu_1(\mathrm{t})$ being the volatility parameter that most strongly reverts and therefore the variable that accounts for the majority of the transitory shocks to volatility. In the CN model for copper, a stronger mean reversion than for oil is found. This is consistent with Schwartz (1997). In the CHS model for copper, it is worth noting that the standard error of futures is similar, in magnitude, to the standard error of futures of the CN model.

Even though Schwartz (1997) found that mean-reverting prices did not seem to hold for gold, our estimates[9] of the TS model being lower than oil and copper estimates, are still significant at standard levels. Also, correlations between the spot price and the forward cost-of-carry curve ($\rho_{13} = -0.2647$, $\rho_{24} = -0.1284$), although small, are significant, contrasting with Casassus and Collin-Dufresne (2005). It is interesting to note that low correlations between the volatility components and the spot price and forward cost-of-carry curve are consistent with the presence of unspanned stochastic volatility in this market, but in a weaker way than for oil and copper. Also, and just as with oil, the first volatility component $\nu_1(t)$ accounts for most of the transitory shocks to volatility and is the most volatile. The estimates for the CN model applied to gold again show some mean reversion, even though weaker than for oil and copper. Finally, the standard deviation for futures of the CHS model for gold is similar to the value reported for the CN, whereas the standard deviation for options is similar to that of the TS model.

Given that the TS model parameters seem to suggest the presence of unspanned stochastic volatility for all three commodities, in order to more formally test for USV, we follow Trolle and Schwartz (2009), and regress straddle returns on each futures contract i:

[8] To our knowledge, this is the first application of the TS model to copper or oil.

[9] It must be noted that we do not use only futures, like Schwartz (1997), but also options in the calibration process.

$$r_t^{straddle,i} = \beta_0^i + \beta_1^i PC_t^{fut,1} + \beta_2^i PC_t^{fut,2} + \beta_3^i PC_t^{fut,3}$$
$$+ \beta_4^i \left(PC_t^{fut,1}\right)^2 + \beta_5^i \left(PC_t^{fut,2}\right)^2 + \beta_6^i \left(PC_t^{fut,3}\right)^2 + \varepsilon_t^i \qquad (32)$$

Table IV presents the results of adjusted R^2, which range from 5.37% to 57.56% for oil, from 2.2% to 80% for copper and from 4.42% to 75.03% for gold. These results suggest that it is very difficult to hedge volatility using only futures contracts, which implies the presence of unspanned stochastic volatility for oil, copper, and gold. We believe this is the first evidence reported in the literature on unspanned stochastic volatility for the copper and gold markets.

4.2. Pricing Performance Comparison

We now compare the pricing performance of the three models. It should be noted that making a fair comparison between models that have different structures sometimes is not straightforward. Someone could argue that for benchmarking purposes, we should hold constant, for example, the number of risk factors, while others could say it is the number of parameters. In order to provide ample information for comparing model performances we calibrate the CN model under three alternative specifications: 5, 4, and 3 risk factors.

For each commodity and model specification we obtain a time series of daily futures and options root mean square pricing errors (RMSE). Errors are defined as the percentage difference between actual and fitted prices, for futures, and as the difference between the fitted and the actual Black and Scholes implied volatilities, for options. For options we present the absolute error and the relative error (divided by the observed Black and Scholes implied volatility), for each maturity.

Results are presented first as the average of daily RMSE for futures and options, for each data Panel. Then a figure with the time series for the errors is shown. Finally, a cross section error analysis for contracts with different maturities, is presented. The analysis is done for each of the three commodities: oil, copper, and gold.

4.2.1. Oil

Table V summarizes the RMSE for Oil futures and options for all specifications and data sets. For futures contracts, 5 and 4-factor CN models perform better than the TS model, whereas for the 3-factor specification there are no significant differences. This is not surprising given that futures prices in the TS model are driven only by 3 factors, instead of the 4 and 5 factors in the CN specifications. The futures pricing errors in Panel A for the TS model are 3.5 and 1.8 times higher than for 5 and 4-factor CN models. The CHS model performs similar to the TS model, which implies a worse behavior than the 5 and 4 factor CN models. These differences are quite stable through the different panels, being a little smaller for the out-of-sample Panel E. Thus for futures pricing, how volatility is modeled appears not to be as relevant as the number of factors considered.

For options contracts, on the other hand, the volatility specification seems to have a great impact. Table V shows that, on average, the TS model substantially outperforms the CHS model and every CN model in all data sets, in and out-of-sample. Also it shows that adding an extra factor to the CN specification does not significantly improve the options pricing performance. For example, in Panel A for the CN model, regardless of the number of factors the RMSE is more than 5 times larger than in the TS model. This difference is even

TABLE IV
R^2s from Straddle Return Regressions

Underlying Future Contract – Oil

Panel	F1 (%)	F2 (%)	F3 (%)	F4 (%)	F5 (%)	F6 (%)	MD1 (%)	MD2 (%)	D1 (%)	D2 (%)	D3 (%)	D4 (%)
2006–2011 (A)	15.24	16.58	18.85	20.49	24.04	25.40	6.39	25.95	26.80	37.79	38.05	43.94
2006–2007 (B)	36.44	37.27	39.75	41.21	45.12	43.21	35.84	39.82	48.12	45.68	50.16	57.56
2008–2009 (C)	12.60	16.85	24.06	28.32	31.79	33.42	5.37	36.44	38.05	47.20	45.90	46.58
2010–2011 (D)	19.38	21.58	22.61	22.25	23.23	24.31	21.74	19.89	7.58	22.09	25.03	34.44

Underlying Future Contract – Copper

Panel	F1 (%)	F2 (%)	F3 (%)	F4 (%)	F5 (%)	F6 (%)	MD1 (%)	MD2 (%)
2006–2011 (A)	7.0	13.6	18.4	17.8	18.4	7.8	34.4	28.8
2006–2007 (B)	10.3	29.8	25.8	24.8	18.7	5.0	35.6	22.8
2008–2009 (C)	7.8	19.1	32.9	24.8	65.6	80.0	59.1	42.8
2010–2011 (D)	7.6	33.3	2.2	10.7	27.6	33.1	–	–

Underlying Future Contract – Gold

Panel	F1 (%)	F2 (%)	F3 (%)	F4 (%)	F5 (%)	F6 (%)	JD1 (%)	JD2 (%)	JD3 (%)	JD4 (%)	JD5 (%)
2006–2011 (A)	4.81	44.86	39.54	38.21	35.68	38.17	23.77	35.10	34.82	27.90	34.84
2006–2007 (B)	19.65	75.03	67.44	60.49	54.75	63.20	42.76	47.65	–	–	–
2008–2009 (C)	4.42	57.38	48.16	51.87	51.54	55.51	31.85	33.98	31.48	–	–
2010–2011 (D)	5.92	64.56	56.02	53.06	51.89	48.40	29.58	38.38	38.61	27.15	33.16

Notes: Adjusted R^2S from regressing straddle returns on the three first principal components and squared principal components of futures returns. Panel A represents the full-sample period, Panels B, C, and D the corresponding subsamples. Fi denotes the first i-month contracts; MDi denotes the i-following contract with expiration either in March, June, September, or December; Di denotes the i-following contract with expiration in December. JDi denotes the first i contracts with expiration either in June or December after a year from date.

TABLE V
Overall RMSE: Oil

	Panel A (2006–2011)	*Panel B* (2006–2007)	*Panel C* (2008–2009)	*Panel D* (2010–2011)	*Panel E* (2012–5'2013)
Futures					
TS	0.35	0.24	0.43	0.29	0.27
5F	0.10	0.08	0.12	0.09	0.12
4F	0.19	0.12	0.25	0.13	0.24
3F	0.34	0.24	0.41	0.31	0.33
CHS	0.37	0.25	0.46	0.32	0.34
Options (Absolute RMSE)					
TS	1.62	0.91	1.64	1.28	1.35
5F	8.50	2.45	10.70	4.68	7.88
4F	8.49	2.46	10.72	4.68	7.85
3F	8.56	2.48	10.67	4.71	7.99
CHS	3.32	1.79	4.06	2.91	2.93
Options (Relative RMSE)					
TS	4.63	3.2	3.95	3.68	4.73
5F	21.41	8.57	21.35	13.44	27.71
4F	21.41	8.6	21.43	13.35	27.65
3F	21.58	8.7	21.33	13.46	27.93
CHS	9.54	6.20	9.77	8.11	9.93

Notes: Average of daily RMSEs for all data sets. RMSE of futures represents the difference between the fitted and the observed price, divided by the observed price. Absolute RMSE of options represents the difference between the fitted and actual Black and Scholes implied volatilities. Relative RMSE represents the difference between the fitted and actual Black and Scholes implied volatilities, divided by the observed Black and Scholes implied volatility. Panel E is the out of sample period between January 2012 and May 2013. Here 5F, 4F, and 3F represent the CN models of 5, 4, and 3 factors, respectively. Errors are expressed in percentages.

larger for the financial crisis period (Panel C), as expected. Despite the fact that the CHS has a greater error than the TS model, it outperforms every CN model in all data sets.

Figure 3 presents the time series of the RMSEs for option contracts for the full sample period, both in and out-of-sample. It can be seen that during the whole period, the TS model outperforms the CHS which, in turn, outperforms the CN models most of the time. The CN performance worsens when volatility increases in the peak of the 2008 financial crisis. This highlights the advantages of considering stochastic volatility.

Finally, Table VI presents a cross section error analysis of implied volatilities for options with different maturities. For all data panels, the Absolute RMSE error differences between the CN and TS models are much greater for shorter than for longer maturity contracts, ranging from more than 6 times higher for the CN model, for the shortest-maturity contract, to almost twice for the longest-maturity contract (around 4 years). Moreover, for periods without a financial crisis, like Panel B and D, the Absolute RMSE for long maturity contracts in both models is similar. Also the CHS model has a greater RMSE than the TS model, but much lower that the CN model. The RMSE differences are reduced as maturity increases. In terms of Relative RMSE the relative errors are lowest for the TS model and highest for the CN model. Relative errors for different maturities behave differently depending on the model, sometimes decreasing with maturity (CN Model[10]) and sometimes decreasing first to later increase for high maturities.

[10]Longer maturity options show less variations in volatility across time and moneyness. This could be the reason why the CN constant volatility model fits better longer maturity options than shorter ones.

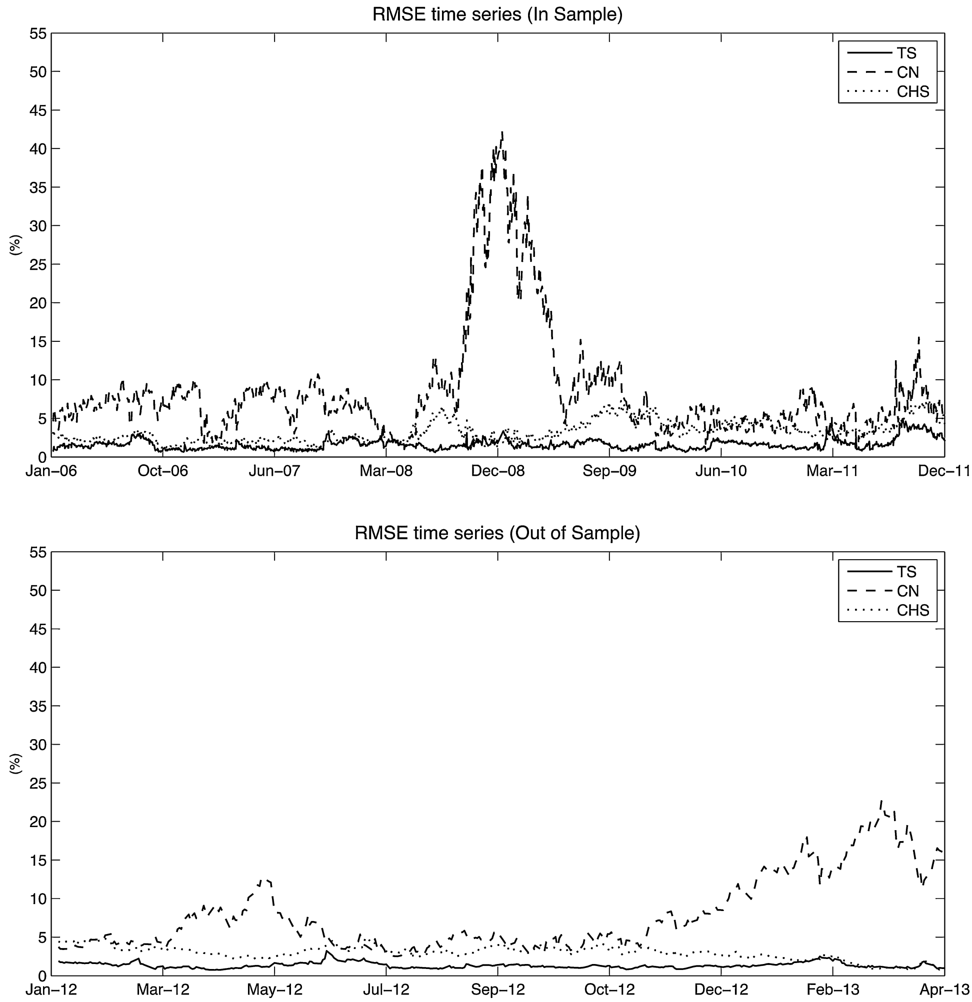

FIGURE 3

RMSE time series: Oil Options (Panel A).

In sample goes from January 2006 to December 2011 and out of sample from January 2012 to May 2013. RMSE of options represents the difference between the fitted and actual Black and Scholes implied volatilities.

4.2.2. Copper

Table VII summarizes the RMSE for Copper futures and options, which are similar to those for the oil case. It can be seen that in futures pricing, CN models tend to outperform the TS and CHS models as the number of factors increases, however all model specifications have small errors ranging from 0.05% to 0.10%. For option contracts, the volatility specification is important, and regardless of the number of factors in the CN models, again the RMSE for Panel A in CN models is more than 5 times larger than in the TS model and more than 2 times larger than the RMSE of CHS model. Like in the oil results, the CN RMSE strongly increases for the financial crisis period (Panel C).

Figure 4 is similar to Figure 3, but now for copper options. It shows that the TS slightly outperforms the CHS model, whereas the CN model behaves much worse, especially during the 2008 financial crisis.

TABLE VI
Cross-Section RMSE: Oil Options

		Panel A			*Panel B*			*Panel C*			*Panel D*			*Panel E*		
		(2006–2011)			*(2006–2007)*			*(2008–2009)*			*(2010–2011)*			*(2012–5'2013)*		
Absolute RMSE																
	τ	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS
F1	0.07	2.38	14.49	4.54	1.22	3.95	2.51	2.10	17.96	5.45	2.38	6.39	4.43	1.73	11.64	4.51
F2	0.15	1.98	12.21	2.72	1.02	3.01	1.78	1.59	14.89	2.32	1.25	6.03	3.22	1.35	10.84	2.25
F3	0.24	1.74	10.63	2.08	0.89	2.62	1.26	1.45	12.87	1.92	1.05	5.63	2.70	1.30	10.10	1.43
F4	0.32	1.47	9.64	2.73	0.73	2.50	1.18	1.39	11.55	3.39	0.94	5.14	2.46	1.28	9.28	2.16
F5	0.40	1.30	9.14	3.52	0.58	2.27	1.53	1.38	10.88	4.71	0.93	4.94	2.47	1.34	9.09	2.80
F6	0.49	1.24	8.14	3.99	0.63	2.18	1.78	1.37	9.85	5.43	0.89	4.70	2.47	1.35	7.82	3.41
MD1	0.64	1.19	7.77	4.57	0.70	2.25	1.95	1.36	8.94	6.49	1.01	4.43	2.63	1.35	7.73	3.77
MD2	0.89	1.34	6.82	4.87	0.89	2.56	2.35	1.38	7.71	6.74	0.92	3.33	2.68	1.46	5.71	4.57
D1	1.45	1.67	5.50	4.93	1.24	2.17	2.43	1.91	4.82	6.31	1.21	3.34	2.71	1.21	5.42	3.65
D2	2.47	1.86	4.23	4.95	1.67	2.29	2.53	2.11	4.49	4.14	1.50	2.35	2.65	1.30	5.40	2.98
D3	3.46	2.24	3.89	5.20	2.52	2.59	2.55	2.32	3.76	2.97	1.78	2.27	3.00	1.53	4.80	2.71
D4	4.39	1.84	3.18	4.81	1.79	2.29	3.11	1.88	4.77	2.48	2.21	2.36	2.68	1.66	4.67	1.71
Relative RMSE																
		TS	CN	CHS	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS
F1	0.07	5.74	26.10	12.18	3.58	11.55	7.76	4.20	28.87	12.19	5.72	16.26	11.81	5.42	51.97	15.78
F2	0.15	5.44	24.82	7.64	3.29	9.21	5.60	3.54	26.60	5.22	3.48	16.38	8.64	4.90	49.21	7.96
F3	0.24	5.00	23.23	5.58	3.00	8.36	4.18	3.38	25.74	4.32	3.06	15.09	6.92	4.94	47.03	4.67
F4	0.32	4.25	22.60	7.18	2.52	8.18	4.10	3.32	26.96	7.48	2.81	13.99	6.30	4.79	43.79	6.57
F5	0.40	3.70	22.01	9.60	2.05	7.77	5.45	3.20	25.81	10.60	2.80	13.31	6.48	5.03	43.31	8.71
F6	0.49	3.46	20.84	11.40	2.23	7.68	6.58	3.28	25.98	12.75	2.65	12.72	6.48	4.90	36.40	10.90
MD1	0.64	3.32	20.73	12.98	2.57	8.42	7.26	3.38	25.64	15.10	3.03	12.18	7.17	4.67	36.01	12.17
MD2	0.89	3.99	19.83	14.45	3.26	9.25	8.90	3.72	23.62	17.35	2.85	10.37	7.77	4.77	25.10	14.42
D1	1.45	5.22	19.42	14.26	4.92	8.94	9.35	5.46	18.22	17.18	4.16	10.84	8.16	4.63	25.01	12.44
D2	2.47	6.87	19.22	14.44	7.83	10.88	10.06	6.77	19.69	13.43	5.73	9.29	8.90	5.77	25.42	10.28
D3	3.46	9.51	19.72	14.73	13.33	13.40	10.00	8.17	15.72	10.64	7.43	9.68	10.25	7.11	22.59	9.91
D4	4.39	7.79	15.28	13.72	8.63	10.12	12.36	7.52	21.56	10.34	9.45	10.34	9.25	8.01	22.20	6.78

Notes: Average of daily RMSEs of the TS and 5-factor CN models across maturity. Absolute RMSE represents the difference between the fitted and actual Black and Scholes implied volatilities. Relative RMSE represents the difference between the fitted and actual Black and Scholes implied volatilities, divided by the observed Black and Scholes implied volatility. Panel E is the out of sample period between January 2012 and May 2013. Fi denotes the first i-month contracts, MDi denotes the i-following contract with expiration either in March, June, September, or December, Di denotes the i-following contract with expiration in December. Maturity of each contract, τ, is expressed in years, errors in percentages.

TABLE VII
Overall RMSE: Copper

	Panel A (2006–2011)	*Panel B* (2006–2007)	*Panel C* (2008–2009)	*Panel D* (2010–2011)	*Panel E* (2012–5'2013)
Futures					
TS	0.09	0.15	0.08	0.05	0.04
5F	0.05	0.08	0.04	0.02	0.02
4F	0.06	0.09	0.05	0.03	0.02
3F	0.09	0.13	0.08	0.05	0.03
CHS	0.10	0.13	0.08	0.05	0.07
Options (Absolute RMSE)					
TS	1.70	2.51	1.25	1.32	1.73
5F	8.76	7.00	10.67	5.16	19.39
4F	8.76	6.94	10.67	5.27	19.40
3F	8.76	6.96	10.68	5.16	19.37
CHS	3.75	3.91	3.83	3.28	9.06
Options (Relative RMSE)					
TS	4.58	6.41	2.86	4.52	7.52
5F	21.35	17.6	22.31	16.68	62.15
4F	21.35	17.39	22.3	17.04	62.18
3F	21.35	17.45	22.32	16.7	62.09
CHS	9.83	9.96	7.54	11.13	43.94

Notes: Average of daily RMSEs for all data sets. RMSE of futures represents the difference between the fitted and the observed price, divided by the observed price. Absolute RMSE of options represents the difference between the fitted and actual Black and Scholes implied volatilities. Relative RMSE represents the difference between the fitted and actual Black and Scholes implied volatilities, divided by the observed Black and Scholes implied volatility. Panel E is the out of sample period between January 2012 and May 2013. Here 5F, 4F, and 3F represent the CN models of 5, 4, and 3 factors, respectively. Errors are expressed in percentages.

Finally, Table VIII presents a cross section error analysis of the implied volatility for copper options with different maturities. Again, the Absolute RMSE and Relative RMSE differences between the three models are much greater for shorter than for longer maturity contracts and for periods of higher financial distress. It is important to note that the longest copper option contract has a maturity of less than a year, so absolute errors averages are higher than those of oil contracts, due to the different average maturity. Finally, like in the oil case, relative errors increase for longer maturity options in stochastic volatility models (TS and CHS models).

4.2.3. *Gold*

Tables IX and X, and Figure 5 repeat for gold the same exercise previously done for oil and copper. Results for gold are very similar to those from the other commodities: the CN model performs better than the TS and CHS model for futures, and much worse for options; during the 2008 financial crisis the CN model behaves particularly bad for options; and finally, the shorter the option contract maturity, the worse the performance of the CN model relative to both stochastic volatility models (TS and CHS model), especially for Panels C, D, and E.

4.3. Implementation Complexity

In addition to the pricing performance just reported, a quantitative analysis of the implementation complexity is proxied by calculating the execution times for each model. To

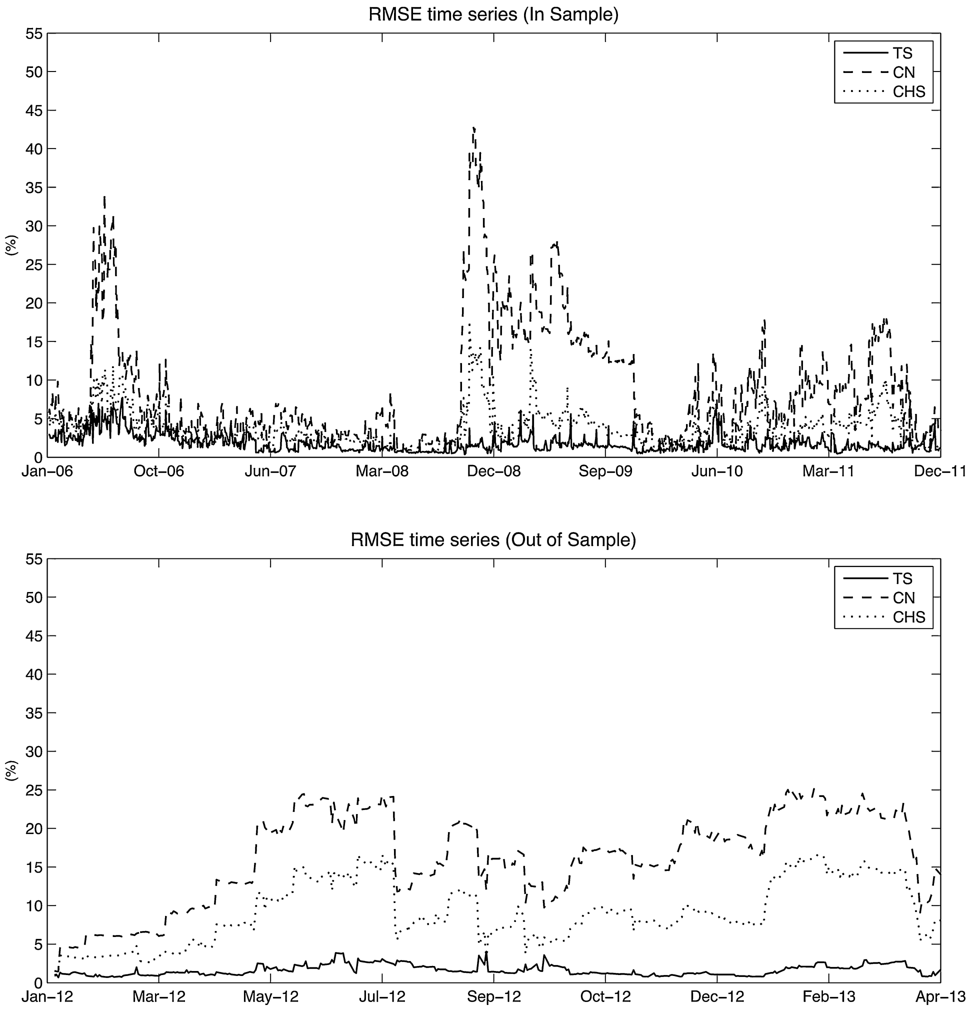

FIGURE 4

RMSE time series: Copper Options (Panel A).

In sample goes from January 2006 to December 2011 and out of sample from January 2012 to May 2013. RMSE of options represents the difference between the fitted and actual Black and Scholes implied volatilities.

make model comparisons, execution times are computed as the time required by each model per iteration of the calibration process.[11] Results reported in the previous section show very clearly that the TS model outperforms in pricing precision the CHS and the CN models, but in this section we are interested on measuring the cost of a better performance in terms of the model implementation complexity.

Table XI shows the execution times for the three models over all data sets.[12] It can be seen that the TS model is between 8 and 15 times slower than the CN specifications. This is

[11]Considering a gradient-base algorithm, to find the optimal solution that uses forward finite differences to approximate the derivatives, an N-parameter model includes $N+1$ valuations of the objective function in each iteration: N valuations for calculating the partial derivatives plus one for valuing the new point.

[12]All measurements are done based on an Intel Core i5, 2.4 GHz processor, 8 GB RAM, OS X system.

TABLE VIII
Cross-section RMSE: Copper Options

		Panel A			*Panel B*			*Panel C*			*Panel D*			*Panel E*		
		(2006–2011)			*(2006–2007)*			*(2008–2009)*			*(2010–2011)*			*(2012–5'2013)*		
		Absolute RMSE														
	τ	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS
F1	0.08	3.3	13.84	5.50	3.26	12.96	6.77	2.41	14.77	4.27	2.82	6.82	4.12	1.46	16.72	6.47
F2	0.17	2.06	11.78	3.62	2.94	10.4	4.35	1.23	12.41	3.13	1.67	6.2	2.80	1.48	17.18	3.53
F3	0.25	2.27	10.39	3.23	2.99	9.66	3.82	1.29	10.64	2.60	1.56	5.56	3.07	1.54	18.18	7.33
F4	0.33	2.14	10.06	4.35	2.61	9.38	4.33	1.77	11.51	4.95	1.49	6.06	3.94	1.52	17.97	10.29
F5	0.42	2.78	10.06	5.32	2.31	8.14	5.08	1.28	13.05	6.23	1.17	5.43	4.50	1.46	17.61	12.16
F6	0.5	1.76	7.97	4.71	2.21	7.74	4.62	1.02	7.99	4.35	1.55	5.99	4.70	1.88	17.55	13.95
MD1	0.65	2.09	8.28	4.88	2.48	8.23	4.70	1.49	7.95	5.11	1.57	6.14	4.56	3.10	15.79	14.66
MD2	0.85	1.96	5.15	4.55	2.49	6.14	4.46	1.08	8.15	5.53	1.84	3.77	3.32	4.31	15.31	16.12
		Relative RMSE														
	τ	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS
F1	0.08	6.73	26.98	12.96	8.21	21.70	15.15	4.15	26.65	7.42	7.21	20.29	13.24	6.25	91.04	27.44
F2	0.17	5.64	27.54	8.82	7.24	21.12	10.41	3.04	23.76	5.74	6.07	21.85	9.51	5.20	91.60	18.85
F3	0.25	5.60	24.83	8.75	7.30	20.91	9.17	2.91	21.70	4.94	4.75	18.63	12.14	5.23	93.87	39.06
F4	0.33	4.86	21.77	10.63	5.94	19.74	9.84	2.46	22.80	8.39	4.42	16.66	12.82	5.81	91.31	53.37
F5	0.42	4.84	23.66	14.60	5.86	19.07	12.11	2.66	22.91	9.98	3.77	18.40	17.90	6.56	87.98	61.39
F6	0.5	5.04	21.03	14.46	6.26	18.89	12.12	3.07	20.24	9.63	5.13	18.59	17.77	10.18	89.63	71.74
MD1	0.65	5.41	18.04	13.04	6.30	17.88	11.67	3.51	18.00	10.44	4.60	16.34	14.44	16.20	77.16	71.86
MD2	0.85	5.79	12.75	13.81	6.94	15.35	12.56	3.36	24.77	16.90	5.81	10.34	10.79	22.45	76.93	80.20

Notes: Average of daily RMSEs of the TS and 5-factor CN models cross maturity. Absolute RMSE represents the difference between the fitted and actual Black and Scholes implied volatilities. Relative RMSE represents the difference between the fitted and actual Black and Scholes implied volatilities, divided by the observed Black and Scholes implied volatility. Panel E is the out of sample period between January 2012 and May 2013. Fi denotes the first i-month contracts; MDi denotes the i-following contract with expiration either in March, May, September, or December. Maturity of each contract, τ, is expressed in years, errors in percentages.

TABLE IX
Overall RMSE: Gold

	Panel A (2006–2011)	*Panel B* (2006–2007)	*Panel C* (2008–2009)	*Panel D* (2010–2011)	*Panel D* (2012–5'2013)
Futures					
TS	0.05	0.04	0.04	0.07	0.03
5F	0.02	0.02	0.02	0.02	0.02
4F	0.03	0.03	0.02	0.03	0.02
3F	0.05	0.04	0.04	0.05	0.04
CHS	0.05	0.04	0.06	0.05	0.03
Options (Absolute RMSE)					
TS	0.92	0.74	0.86	0.75	1.21
5F	5.17	3.68	5.90	3.23	5.28
4F	5.16	3.70	5.90	3.52	5.26
3F	5.08	3.72	5.86	3.27	5.15
CHS	1.91	2.04	2.09	1.70	1.59
Options (Relative RMSE)					
TS	3.74	3.45	2.93	3.05	5.77
5F	20.69	18.13	18.83	14.81	27.73
4F	20.67	18.23	18.87	16.06	27.69
3F	20.5	18.13	18.82	14.92	27.32
CHS	7.76	9.43	6.95	7.16	7.93

Notes: Average of daily RMSEs for all data sets. RMSE of futures represents the difference between the fitted and the observed price, divided by the observed price. Absolute RMSE of options represents the difference between the fitted and actual Black and Scholes implied volatilities. Relative RMSE represents the difference between the fitted and actual Black and Scholes implied volatilities, divided by the observed Black and Scholes implied volatility. Panel E is the out of sample period between January 2012 and May 2013. Here 5F, 4F, and 3F represent the CN models of 5, 4, and 3 factors, respectively. Errors are expressed in percentages.

due to the lack of closed form solutions, which requires the implementation of numerical methods for evaluating option pricing formulas. For illustration purposes on the effect that this could have, if we consider that on average each starting point takes about 15 iterations to get to an optimum and we use a grid of 100 different starting points to maximize the probability of reaching a global optimum, the process would take in a standard system around 8.3, 4.3, or 6.2 days to run TS model on the data sample for oil, copper or gold, respectively. Note that on the same computer, the CN model would take only 0.71, 0.59, or 0.69 days, respectively, depending on the commodity. Thus, in order to improve the 5 times larger errors of the CN model on short-term options if the TS model is used, 10 times more effort is required. The CHS model has a similar execution time as a CN 3F model for copper and gold and as a CN 4F model for oil. Compared to the TS model, the CHS is almost 10 times faster, which is relevant considering that both models account for stochastic volatility.

5. CONCLUSIONS

Prices of commodity contingent claims depend on the process assumed for the underlying asset. For futures, a good specification of the drift is very important, but for options, the volatility specification is crucial. Roughly speaking, commodity models in the literature can be classified into two kinds: those with a constant volatility and those with a stochastic volatility specification. In this article, these two ways of dealing with volatility are compared in a manner

TABLE X
Cross-Section RMSE: Gold Options

		Panel A			*Panel B*			*Panel C*			*Panel D*			*Panel E*		
		(2006–2011)			*(2006–2007)*			*(2008–2009)*			*(2010–2011)*			*(2012–5'2013)*		
		Absolute RMSE														
	τ	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS
F1	0.08	1.74	6.67	3.32	1.18	4.48	2.52	1.18	8.36	3.75	1.77	4.69	2.99	2.52	7.36	2.96
F2	0.20	1.22	6.66	2.29	0.87	4.51	2.16	0.92	7.63	2.39	1.11	4.23	2.13	1.74	7.41	2.13
F3	0.36	0.88	6.55	1.29	0.72	4.21	1.67	0.81	7.05	1.36	0.65	3.81	1.24	1.25	6.68	1.42
F4	0.53	0.77	6.35	1.29	0.65	4.41	1.88	0.71	6.44	1.66	0.51	3.59	1.21	0.95	5.89	1.09
F5	0.69	0.67	6.12	1.73	0.61	4.32	2.25	0.65	6.06	2.12	0.46	3.45	1.44	0.79	5.39	1.10
F6	0.85	0.64	6.22	2.23	0.68	4.54	2.56	0.68	5.95	2.42	0.46	3.41	1.68	0.72	5.11	1.28
JD1	1.49	0.89	5.17	3.24	0.86	4.47	3.15	1.29	5.33	4.14	0.51	2.83	1.93	0.72	3.45	1.79
JD2	2.00	0.89	4.60	3.09	1.22	4.45	3.54	1.93	6.88	4.44	0.69	2.87	2.09	0.77	2.85	1.91
JD3	2.48	0.79	2.96	2.82	–	–	–	1.84	4.43	1.40	0.91	2.44	2.06	0.77	1.72	1.91
JD4	3.03	0.75	3.24	2.98	–	–	–	0.77	1.59	2.02	1.12	1.98	1.48	–	–	–
JD5	3.53	1.47	7.54	5.95	–	–	–	1.77	5.37	5.53	1.41	1.55	1.33	–	–	–
		Relative RMSE														
	τ	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS	TS	CN	CHS
F1	0.08	6.63	29.44	13.98	4.92	21.68	10.95	3.89	30.17	13.46	7,06	24,53	13.72	10,43	49,13	13.70
F2	0.20	4.99	27.69	9.75	4.03	21.1	9.51	3.18	24.7	8.53	4,61	20,33	9.46	8,39	48,83	10.79
F3	0.36	3.71	25.86	5.30	3.35	19.92	7.63	2.79	20.57	4.41	2,86	17,03	5.56	6,6	42,84	7.11
F4	0.53	3.3	25.85	4.96	3.08	21.57	8.83	2.4	18.86	5.13	2,09	15,07	5.19	5,01	36,79	5.64
F5	0.69	2.82	24.24	6.76	2.74	20.92	10.31	2.3	17.95	6.43	1,9	13,78	5.86	4,18	32,95	6.00
F6	0.85	2.62	24.76	8.59	2.96	21.93	11.57	2.36	17.41	7.29	1,8	13,13	6.49	3,8	30,76	7.00
JD1	1.49	4.02	20.55	11.82	4.28	21.73	13.86	4.72	17.36	12.21	2	10,55	7.13	3,62	18,98	8.92
JD2	2.00	3.61	15.39	11.11	5.7	19.92	13.99	6.99	20.91	12.84	2,55	10,04	7.36	3,61	14,52	8.56
JD3	2.48	3.03	10.41	10.18	–	–	–	7.39	18.05	5.79	3,57	9,08	7.50	3,07	6,37	7.24
JD4	3.03	2.86	11.37	10.64	–	–	–	2.82	5.91	7.02	4,47	8,18	6.05	–	–	–
JD5	3.53	4.91	21.37	17.42	–	–	–	5.89	15.63	15.52	5,54	6,11	5.04	–	–	–

Notes: Average of daily RMSEs of the TS and 5-factor CN models cross maturity. Absolute RMSE represents the difference between the fitted and actual Black and Scholes implied volatilities. Relative RMSE represents the difference between the fitted and actual Black and Scholes implied volatilities, divided by the observed Black and Scholes implied volatility. Panel E is the out of sample period between January 2012 and May 2013. Fi denotes the first i-month contracts; JDi denotes the first i contracts with expiration either in June or December after a year from date. Maturity of each contract, τ, is expressed in years, errors in percentages.

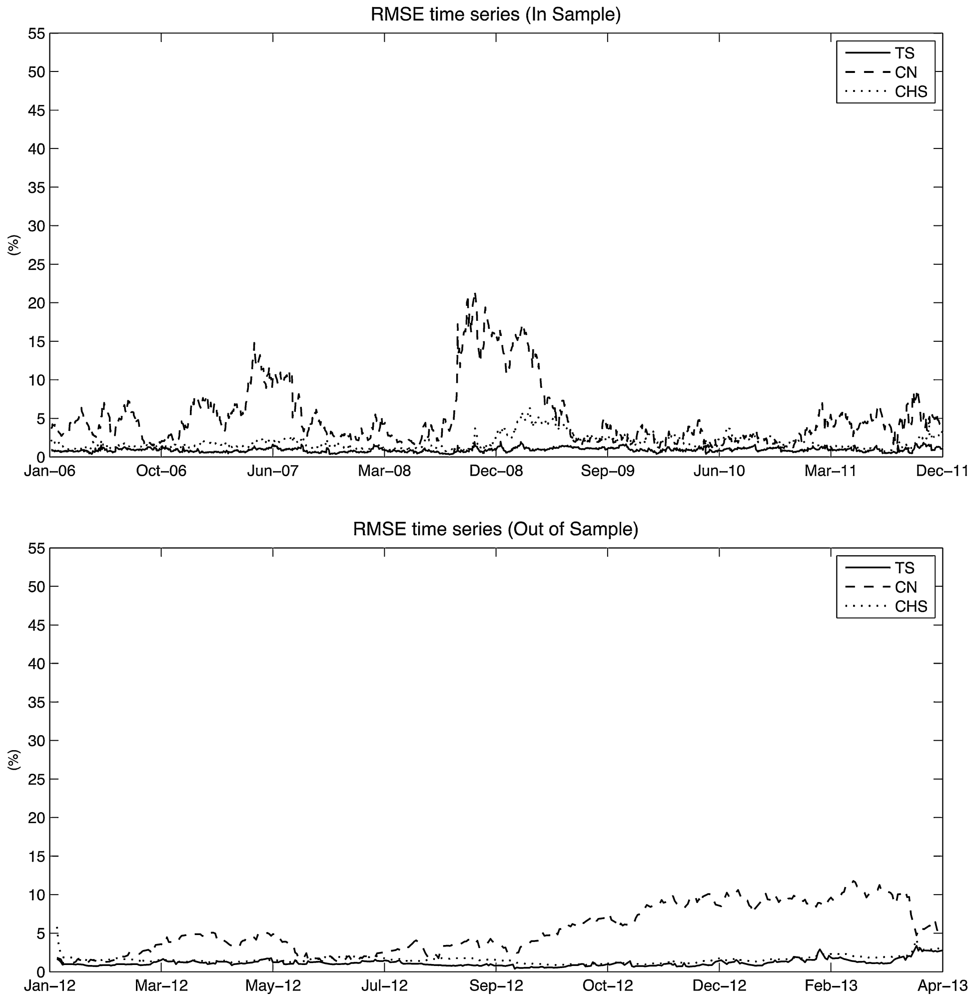

FIGURE 5

RMSE time series: Gold Options (Panel A).

In sample goes from January 2006 to December 2011 and out of sample from January 2012 to May 2013. RMSE of options represents the difference between the fitted and actual Black and Scholes implied volatilities.

relevant to practitioners eliciting the tradeoffs between the empirical performance and the implementation effort required for each model and commodity contract.

To make this comparative analysis three models are chosen: the Cortazar and Naranjo (2006) CN model represents the constant volatility specification, the Trolle and Schwartz (2009) TS model represents the classic stochastic volatility model, and the Chiang et al. (2015) CHS model represents a new stochastic volatility model with a model-free implied variance specification. These models are then applied to oil, copper and gold, to both futures and options data during different time periods. Pricing errors are calculated and execution times measured.

Results for all commodities are, in general, consistent. First, for futures pricing it is clearly better to use the CN model, because not only is it simpler, but also errors are smaller. Also the higher the number of risk factors in the model, the smaller the errors. Second, option

TABLE XI
Execution Times per Iteration

	Panel A (2006–2011)	*Panel B* (2006–2007)	*Panel C* (2008–2009)	*Panel D* (2010–2011)
Oil				
TS	477.16	228.71	220.11	200.20
5F	59.44	18.66	21.07	20.62
4F	41.03	13.50	14.46	14.44
3F	29.02	9.50	9.51	9.76
CHS	48.57	14.37	16.92	16.61
Copper				
TS	247.39	179.64	160.86	107.88
5F	45.54	15.38	18.38	17.02
4F	34.57	11.55	11.74	13.22
3F	27.86	7.25	6.49	8.88
CHS	24.66	8.18	7.76	7.83
Gold				
TS	358.53	164.34	203.52	271.32
5F	55.84	19.17	18.48	21.35
4F	40.70	11.04	12.58	14.56
3F	22.66	6.87	8.40	8.53
CHS	27.62	8.35	8.82	9.60

Notes: Execution times are calculated based on $N+1$ valuations of the objective function. Times are expressed in seconds per iteration. Here 5F, 4F, and 3F represent the CN models of 5, 4, and 3 factors, respectively.

pricing errors are considerably higher using the CN model, increasing at the most by a factor of 6. Third, the longer the option maturity, the less relevant is the difference in pricing errors. Fourth, the CHS model works as an intermediate model, showing a good pricing of futures and an option pricing better than the CN model and slightly worse that the TS model. Given its much simpler implementation, and better execution times, it presents a great alternative as a stochastic volatility model. Fifth, the TS model is much more complex to implement and its execution times are about 10 times higher than for the CN model. Sixth, our results of implementing the TS model for copper and gold are consistent with unspanned stochastic volatility for both commodities.

Results presented in this article are new and relevant for practitioners. Up to now it is, to our knowledge, the first work to empirically test the pricing performance, using futures and options contracts, of stochastic volatility models against constant volatility benchmarks for oil, copper and gold. Also it is the first to apply the TS model and the CHS model to copper and gold markets. Choosing the best model to implement in a real situation depends on the objectives pursued and on the tradeoffs between effort and precision.

APPENDIX A: MODELS DETAILS

A.1. Affine transformation of the HJM process

Given the specification of $\sigma_{yi}(t,T), i = 1,2$, and $\mu_y(t,T)$ described in section 2.1, the t-time instantaneous forward cost of carry at time T, $y(t,T)$, is given by

$$y(t,T) = y(0,T) + \sum_{i=1}^{2}\left(\alpha_i e^{-\gamma_i(T-t)}x_i(t) + \alpha_i e^{-2\gamma_i(T-t)}\phi_i(t)\right) \tag{33}$$

where $x_i(t)$ and $\phi_i(t)$, $i = 1,2$, evolve according to following system of differential equations:

$$dx_1(t) = \left(-\gamma_1 x_1(t) - \left(\frac{\alpha_1}{\gamma_1} + \rho_{13}\sigma_{S1}\right)\nu_1(t)\right)dt + \sqrt{\nu_1(t)}dW_3^Q(t) \tag{34}$$

$$dx_2(t) = \left(-\gamma_2 x_2(t) - \left(\frac{\alpha_2}{\gamma_2} + \rho_{24}\sigma_{S2}\right)\nu_2(t)\right)dt + \sqrt{\nu_2(t)}dW_4^Q(t) \tag{35}$$

$$d\phi_i(t) = \left(-2\gamma_i\phi_i(t) + \frac{\alpha_i}{\gamma_i}\nu_i(t)\right)dt, i = 1,2 \tag{36}$$

Subject to $x_i(0) = \phi_i(0) = 0$, $i = 1,2$.

A.2. Transform equations (Trolle & Schwartz, 2009)

To price options on futures, a transform of $F(t,T)$ is introduced in Section 2.1

$$\Psi(u,t,T_0,T_1) = E_t^Q\left[e^{ulog(F(T_0,T_1))}\right] \tag{37}$$

This transform has an affine solution given by

$$\Psi(u,t,T_0,T_1) = e^{M(T_0-t)+N_1(T_0-t)\nu_1(t)+N_2(T_0-t)\nu_2(t)+ulog(F(t,T_1))} \tag{38}$$

where $M(\tau)$, $N_1(\tau)$, and $N_2(\tau)$ solve the following system of ordinary differential equations

$$\frac{dM(\tau)}{d\tau} = N_1(\tau)\eta_1 + N_2(\tau)\eta_2 \tag{39}$$

$$\begin{aligned}\frac{dN_1(\tau)}{d\tau} &= -N_2(\tau)\kappa_{21} + N_1(\tau)\left(-\kappa_1 + u\sigma_{\nu 1}\left(\rho_{15}\sigma_{S1} + \rho_{35}\frac{\alpha_1}{\gamma_1}\left(1 - e^{-\gamma_1(T_1-t)}\right)\right)\right)\\ &+ \frac{1}{2}N_1(\tau)^2\sigma_{\nu 1}^2 + \frac{1}{2}\left(u^2 - u\right)\left(\sigma_{S1}^2 + \left(\frac{\alpha_1}{\gamma_1}\left(1 - e^{-\gamma_1(T_1-t)}\right)\right)^2 + 2\rho_{13}\sigma_{S1}\frac{\alpha_1}{\gamma_1}\left(1 - e^{-\gamma_1(T_1-t)}\right)\right)\end{aligned} \tag{40}$$

$$\begin{aligned}\frac{dN_2(\tau)}{d\tau} &= -N_1(\tau)\kappa_{12} + N_2(\tau)\left(-\kappa_2 + u\sigma_{\nu 2}\left(\rho_{26}\sigma_{S2} + \rho_{46}\frac{\alpha_2}{\gamma_2}\left(1 - e^{-\gamma_2(T_1-t)}\right)\right)\right)\\ &+ \frac{1}{2}N_2(\tau)^2\sigma_{\nu 2}^2 + \frac{1}{2}\left(u^2 - u\right)\left(\sigma_{S2}^2 + \left(\frac{\alpha_2}{\gamma_2}\left(1 - e^{-\gamma_2(T_1-t)}\right)\right)^2 + 2\rho_{24}\sigma_{S2}\frac{\alpha_2}{\gamma_2}\left(1 - e^{-\gamma_2(T_1-t)}\right)\right)\end{aligned} \tag{41}$$

subject to the boundary conditions $M(0) = N_1(0) = N_2(0) = 0$.

A.3. Transform equations (Chiang et al., 2015)

Following Trolle and Schwartz (2009), we introduce the transform:

$$\Psi(u,t,T_0,T_1) = E_t^Q\left[e^{ulog(F(T_0,T_1))}\right] \tag{42}$$

This transform has an affine solution given by

$$\Psi(u,t,T_0,T_1) = e^{M(T_0-t)+N(T_0-t)X_{v,t}+ulog(F(t,T_1))} \tag{43}$$

where $M(\tau)$ and $N(\tau)$ solve the following system of ordinary differential equations[13]

$$\frac{dM(\tau)}{d\tau} = \frac{1}{2}\left(u^2-u\right)\eta^T(T_1-t)\Omega_0\eta(T_1-t)+N(\tau)\mu_2 \tag{44}$$

$$\frac{dN(\tau)}{d\tau} = \frac{1}{2}\left(u^2-u\right)\eta^T(T_1-t)\Omega_1\eta(T_1-t)+N(\tau)[-\kappa_3+u\eta_3(T_1-t)\rho_2\sigma_3]+\frac{N^2(\tau)}{2}\sigma_3^2 \tag{45}$$

subject to the boundary conditions $M(0)=N(0)=0$.

APPENDIX B: ESTIMATION DETAILS

For estimation and calibration of the parameters, the Kalman filter (KF) is applied with the method of maximum likelihood (ML). This requires translating the CN and TS dynamics to their state-space representation. This is accomplished defining the relationship between the state variables of the system and the observed price vector of futures and options (Measurement Equation), and discretizing the dynamics of the state variables (Transition Equation). In particular, for the TS specification, given the nonlinearity of the USV model, the extended version of the Kalman filter (EKF) is used, which linearizes the Measurement Equation, and applies the method of quasi-maximum likelihood (QML) for the parameters calibration using a Gaussian distribution to approximate the true distribution of the innovation errors.

The KF is a widely used estimation methodology (Cortazar & Naranjo, 2006; Cortazar, Schwartz, & Naranjo, 2007; Cortazar, Schwartz, & Tapia, 2012; Pindyck, 2004; Schwartz, 1997; Schwartz & Smith, 2000; Richter & Sørensen, 2002; Trolle & Schwartz, 2009) that calculates, recursively, optimal estimates of unobservable variables using all past information. Then, parameter estimates can be obtained by maximizing the likelihood function of its innovation errors.

In order to apply the KF, models have to be expressed in their state-space representation. The first step is to relate the vector of observables variables, options and futures prices, z_t, to the vector of state variables, X_t. Let X_t^{TS} and X_t^{CN} be the vector of states variables of the TS and CN specification, respectively, and let h^M be the functional form that summarizes the pricing formulas of model M then

$$z_t = h^M\left(X_t^M\right)+u_t, u_t \sim \text{iid } N(0,\Omega) \tag{46}$$

where z_t is a vector of $m_t \times 1$ observations that may vary through time, X_t^M denotes the vector of state variables of model M, and

[13]Here $\eta=(\eta_s,\ \eta_1,\ \eta_p,\ 0)$.

$$X_t^{TS} = (s(t), x_1(t), x_2(t), \phi_1(t), \phi_2(t), \nu_1(t), \nu_2(t)) \tag{47}$$

$$X_t^{CN} = (x_1(t), \cdots, x_N(t)) \tag{48}$$

The measurement Equation (46) requires the existence of a linear relation between observed variables and the state variables and since nonlinear options prices are considered in the observations vector, z_t, the h-function must be linearized.[14] Let $\hat{X}_{t|s}^M = E_s[X_t^M]$ the expectations of X_t including the information until z_s. Then we have

$$z_t = c_t^M + H^M X_t^M + u_t, u_t \sim iidN(0, \Omega_t^M) \tag{49}$$

where $c_t^M = \left(\left(h^M \hat{X}_{t|t-1}^M\right) - H^M \hat{X}_{t|t-1}^M\right)$ and

$$H^M = \frac{\partial h^M (X_t^M)}{\partial X_t^M} |_{X_t^M = \hat{X}_{t|t-1}^M} \tag{50}$$

The transition equation describes the stochastic process followed by the states variables and it can be obtained from the risk-neutral dynamic along with the market price of risk specifications described in section 2.1 and 2.2 for each model:

$$\begin{gathered} X_{t+1}^M = \Phi_0^M + \Phi_X^M X_t^M + \omega_{t+1}^M, \omega_{t+1}^M \sim \text{iid} \\ E\left[\omega_{t+1}^M\right] = 0 \\ Var\left[\omega_{t+1}^{TS}\right] = Q_0^{TS} + Q_{\nu_1}\nu_1(t) + Q_{\nu_2}\nu_2(t) \\ Var\left[\omega_{t+1}^{CN}\right] = Q_0^{CN} \end{gathered} \tag{51}$$

where Φ_0^M, Φ_X^M, Q_0^M, Q_{ν_1}, and Q_{ν_2} can be computed in closed form following Fisher and Gilles (1996).

The KF recursively calculate the optimal estimates of $\hat{X}_t^M$ and the variance–covariance matrix $P_t^M = E[(X_t^M - \hat{X}_t^M)(X_t^M - \hat{X}_t^M)']$ by minimizing the prediction error, $\epsilon_t^M = (z_t - \hat{Z}_{t|t-1}^M)$, in each step. Given $\hat{X}_{t-1}^M$ and P_{t-1}^M, the first step is to compute the predictions at time t for the states variables, $\hat{X}_{t|t-1}^M$, and for the variance-covariance matrix, $P_{t|t-1}$, given all the information up to time t–1:

$$\hat{X}_{t|t-1}^M = \Phi_0^M + \Phi_X^M \hat{X}_{t-1}^M \tag{52}$$

$$P_{t|t-1}^M = \Phi_{X_M} P_{t-1}^M \Phi_X^{M\prime} + Var[\omega_t^M] \tag{53}$$

Then predictions on the observed variables are made and prediction or innovation errors ϵ_t^M, along with their associated variance-covariance matrix, F_t^M, are calculated:

[14]Note that only the TS model is linearized since options are priced based on the actual, rather than the fitted, futures prices. As a consequence, CN option prices do not depend directly on the state variables and the TS specification only on $v_1(t)$ and $v_2(t)$.

$$\widehat{Z}_{t|t-1} = h^M\left(\hat{X}^M_{t|t-1}\right) \tag{54}$$

$$\epsilon^M_t = \left(z_t - h^M\left(\hat{X}^M_{t|t-1}\right)\right) \tag{55}$$

$$F^M_t = H^M_t P_{t|t-1} H^{M\prime}_t + \Omega^M_t \tag{56}$$

This is what is known as the prediction step in the KF. Once the prediction step is conducted, it follows the update step where optimal solutions for the state variables vector and the variance-covariance matrix are calculated:

$$\hat{X}^M_t = \hat{X}^M_{t|t-1} + P^M_{t|t-1} H^{M\prime}_t F^{M-1}_t \epsilon^M_t \tag{57}$$

$$P^M_t = P^M_{t|t-1} + P^M_{t|t-1} H^{M\prime}_t F^{M-1}_t H^M_t P^M_{t|t-1} \tag{58}$$

The estimation of the model parameters, Θ^M, is obtained by maximizing the log-likelihood function of innovations:[15]

$$logL(\Theta^M) = \frac{1}{2}\sum_t^T log|F^M_t| - \frac{1}{2}\sum_t^T \varepsilon^{M\prime}_t F^{M-1}_t \varepsilon^M_t \tag{59}$$

where T is number of observation dates and Θ^M is the vector of unknown parameters, of model M, to be estimated.

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[15]For the CN model the innovation distribution is Gaussian provided by the fact that futures are log-normally distributed. The distribution for the TS innovation is not Gaussian, as its variance-covariance matrix depends on the volatility factors, $v_1(t)$, but is approximated by it. This is what is known as the QML method.

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